Research Statement

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Overview

My research centers in low dimensional topology. In particular I am interested in contact geometry, and Heegaard Floer homologies. I have proved results concerning knots in contact 3–manifolds and studied applications of Heegaard Floer homology to contact 3–manifolds [27, 29]. I will continue my research in this vein, and extend my repertoire with other tools, such as convex surface theory and contact homology.

Goals

Classify Legendrian knots of some knot types. An ongoing research of mine concerns classification of Legendrian representatives of positive braids, braids with few (≤ 3) strands, twist knots and maybe of two bridge knots. I also would like to obtain upper bounds for the number of Legendrian representatives for a fairly wide class of other knot types.

Compute Heegaard Floer homologies of negative definite plumbings. I would like to get a better understanding of plumbings with trivial Heegaard Floer homologies. For instance, does the Heegaard Floer homology remains trivial after deleting a vertex from the corresponding graph? This question was already researched and tested for many graphs by others [17, 14].

Understand relations between Legendrian invariants in Heegaard Floer theories. Using the language of Heegaard Floer homology recently three different invariants were defined for Legendrian and transverse knots [9, 12, 25]. With Stipsicz we understood the connection between two of them [27]. Even though the third behaves similarly, its connection with the other two is yet to be understood.

Characterize contact 3–manifold with vanishing contact invariant. The contact invariant in Heegaard Floer homology turned out to be extremely useful in determining properties such as fillability or tightness of contact structures. In particular it was proved that it vanishes for overtwisted contact structures [22] (and even for ones with Giroux torsion [7]), and is nonzero for Stein-fillable contact structures. The goal would be to understand this gap better.

Research Summary and Background

Heegaard Floer theories. *Heegaard Floer homologies*, (Ozsváth-Szabó, [20, 21, 23]) the recently-discovered invariants for 3- and 4-manifolds, come from an application of Lagrangian Floer homology to spaces associated to Heegaard diagrams. Although this theory is conjecturally isomorphic to Seiberg-Witten theory, it is more topological and combinatorial in its flavor and thus easier to work with in certain contexts. These homologies admit generalizations and refinements for knots (Ozsváth-Szabó [19] and Rasmussen [26]) and links (Ozsváth-Szabó [24]) in 3–manifolds and for non-closed 3–manifolds with certain boundary conditions (Juhász [10]), called sutured Floer homology. The tools used to define the link-version were later applied to define a completely combinatorial version of knot Floer homology in the 3–sphere.

Contact 3–manifolds. Although contact geometry was born in the late 19th century in the work of Sophus Lie, it has just recently started to develop rapidly, with the discovery of convex surface theory and by recognizing their role in other parts of topology. For example Property P for knots —a possible first step for resolving the Poincaré conjecture— was proved using contact 3–manifolds (Kronheimer-Mrowka [11]). Also, the fact that Heegaard Floer homology determines the Seifert genus of a knot was first proved with the help of contact 3–manifolds (Ozsváth-Szabó [18]). Being the natural boundaries of Stein domains, the use of contact 3–manifolds resulted in a topological description of Stein-manifolds. A *contact structure* on an oriented 3–manifold is a totally non-integrable plane field. In other words it is a plane distribution that is not everywhere tangent to any open embedded surface. Any 3–manifold admits a contact structure (Martinet [13]). It is more subtle though to understand the set of all different contact structures on a given 3–manifold. One way to understand them is by examining lower dimension submanifolds that respect the structure in a way. The 2 dimensional such submanifolds are called *convex surfaces*. These are surfaces with a vectorfield in their neighborhood which is transverse to the surface and whose flow preserves the contact plane distribution. Contact structures in the neighborhood of a convex surface are determined by a set of closed curves (*dividing curves*) on the surface (Giroux [8]). Thus convex surfaces became the right boundary conditions for contact 3–manifolds. In Heegaard Floer homology contact invariants were defined for contact 3–manifolds without (Ozsváth-Szabó [22]) or with (Honda-Kazez-Matic [9]) boundary. These invariants had many applications the most recent is a new proof for the fact that a contact 3–manifold having Giroux torsion cannot be Stein-fillable (Ghiggini-Honda-Van Horn-Morris [7]).

Legendrian and transverse knots. There are two ways for a one dimensional submanifold to respects the contact structure. Its tangents can entirely lie in the plane distribution, in which case the knot is called *Legendrian knot*, or if the tangents are transverse to the planes, the knot is then called a *transverse knot*. A Legendrian knot with a given knot type has two classical invariants: its Thurston-Bennequin number and its rotation number. While for transverse knots there is only one invariant; the self-linking number. The problem of classifying Legendrian (transverse) knots up to Legendrian (transverse) isotopy naturally leads to the question whether these invariants classify Legendrian (transverse) knots. A knot type is called *Legendrian (transverse) simple* if any two realizations of it with equal classical invariants are Legendrian (transverse) isotopic. The unknot (Eliashberg-Fraser [3]), torus knots and the figure-eight knot (Etnyre-Honda [6]) were proved to be both Legendrian and transversely simple. By constructing a new invariant for Legendrian knots, Chekanov [2] showed that not all knots are Legendrian simple, in particular he proved that the knot 5_2 is not Legendrian simple. Later many other Legendrian non-simple knots were found (Epstein-Fuchs-Meyer [4] and Ng [15]). The case for transverse knots turned out to be harder. Birman and Menasco [1], and Etnyre and Honda [5] constructed families of transversely non-simple knots using braid and convex surface theory. The Legendrian invariant in the combinatorial Floer homology provided another tool to construct transversely non-simple knots (Ng-Ozsváth-Thurston [16]) By proving a connected sum formula for the combinatorial Legendrian invariant, I proved, the existence of infinitely many transversely non-simple knots:

Theorem 3.1 (Vértesi [29])**.** *There exist infinitely many transversely non-simple knots.*

The definition of the contact invariant in Heegaard Floer homology admits a generalization for Legendrian and transverse knots $\widehat{\mathcal{L}}$ in the knot Floer homology (Lisca-Ozsváth-Stipsicz-Szabó [12]). The contact invariant of Honda, Kazez and Matic for the complement of a Legendrian knot gives rise to a Legendrian invariant: the EH-class. With Stipsicz we understood the relation between these two invariants:

Theorem 3.2 (Stipsicz-Vértesi [27])**.** *There is a map from the sutured Floer homology for the knot-complement to the knot Floer homology mapping $\widehat{\mathcal{L}}$ to* EH*.*

A nice consequence of this theorem, which was independently obtained by Vela-Vick [28], is the following:

Theorem 3.3 (Stipsicz-Vértesi [27])**.** *If the knot complement contains Giroux torsion, then $\widehat{\mathcal{L}}$ vanishes.*

Research Plan

Heegaard Floer homologies of negative definite plumbings. *Plumbings* are spaces that can be described combinatorially. For every weighted graph one can associate a manifold, which is a collection of thickened spheres corresponding to the vertices glued together according to the edges. Despite the easy description, their Heegaard Floer invariants are yet to be understood. Ozsváth and Szabó gave an algorithm to compute Heegaard Floer homologies of plumbings associated to plumbing trees not containing "bad" vertices [17], Némethi extended their result for a wider class of negative definite plumbings [14]. A different kind of approach, would be an inductive description of the homologies. In this direction the first essential step is to investigate how the Heegaard Floer homologies of a plumbing change after deleting a vertex from the corresponding graph. I want to prove that whenever the Heegaard Floer homologies are trivial for a plumbing then it remains trivial for the plumbing obtained by deleting a vertex of the corresponding graph. I want to attack this problem by a tricky application of the exact triangle of Heegaard Floer homology for surgeries along knots. This problem was first conjectured by András Némethi, and originates in singularity theory. He also checked it for some classes of plumbings, for which the statement holds.

Heegaard Floer homologies and contact 3–manifolds. It was known [22] that the contact invariant is non-zero for Stein fillable contact structures and zero for overtwisted ones. My goal is to understand its behavior for some tight but non-Stein fillable manifolds. The contact invariant for contact 3–manifolds with boundary gave a new way to attack this question, by the observation, that the contact invariant vanishes whenever the contact 3–manifold contains a contact submanifold with vanishing invariant (Honda-Kazez-Matic [9]). Since then it was proved that the contact invariant vanishes for contact 3–manifolds containing Giroux torsion [7]. As a first step I want to give a wide class of "small" contact manifolds with boundary with vanishing invariant, and use them as a criteria. The most optimistic idea would be to find a complete list (if there is one) of these submanifolds that can cause vanishing.

Legendrian and transverse knots. The usual way of classifying Legendrian representatives of a knot type consist of three steps. The first one is to prove, that any knot with non-maximal Thurston-Bennequin number is gotten from one with maximal Thurston-Bennequin number by a sequence of well-understood operations called stabilizations. This is not true for any knot type (Etnyre-Honda [5]), and can be subtle to prove. The second step is to understand the maximal Thurston-Bennequin representatives, and at last one needs to understand the relation between the stabilizations of the maximal Thurston-Bennequin representatives. The only transversally non-simple knot type with a complete classification is the $(2,3)$-cable of the $(2,3)$ torus knot (Etnyre-Honda [5]). Using convex surface theory recently with J. Etnyre we managed to understand Legendrian representations of open braids. These techniques should allow us to give a complete classification of Legendrian representatives of positive braids, braids with few (≤ 3) strands. This idea on its own can only be used for knots satisfying the first condition, and as it cannot distinguish Legendrian knots it can only give an upper bound for knot types that are non Legendrian simple. For a complete classification one needs to use other tools as well. Using contact homology and Heegaard Floer homology with Ng we hope to give a complete classification of twist (aka. Chekanov) knots and maybe of two bridge knots.

Using the language of Heegaard Floer homology recently three different invariants were defined for Legendrian and transverse knots. One in the combinatorial settings of knot Floer homology for the 3–sphere [25]: $\widehat{\lambda}$, one in knot Floer homology for a general contact 3–manifold [12]: $\widehat{\mathcal{L}}$ and one defined as the contact invariant associated to the knot-complement: EH. With András Stipsicz [27] we understood the connection between the last two of them; there is a map between the homologies sending EH to $\widehat{\mathcal{L}}$. This suggests, that EH contains more information about a Legendrian knot than $\widehat{\mathcal{L}}$. Morally EH includes all surgery information of the knot. However there is no known examples, that can be distinguished by the EH-class but not by $\widehat{\mathcal{L}}$. The 5_2 knot seems to be a good candidate for proving the difference of these invariants. In the standard contact 3–sphere the first two invariants, though behave fairly similarly; both of them is in the knot Floer homology, for an unoriented knot there are naturally two of each, they vanish under the same kind of operations, etc. Thus it is conjectured that for knots in the 3–sphere the combinatorial invariant equals $\widehat{\mathcal{L}}$. Both of them have a concrete description; $\widehat{\lambda}$ is defined through grid diagrams on the torus, while $\widehat{\mathcal{L}}$ is described using an open book of the standard contact structure, with the knot being on its page. The proof of the equality should go by finding an open book that is related to the toroidal grid diagram by a "nice" sequence of Heegaard moves, under which the transformation of the invariant can be tracked.

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