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Transversely Non-Simple Knots

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By proving a connected sum formula for the Legendrian invariant λ_+ in knot Floer homology we exhibit infinitely many transversely non-simple knot types.

57M27, 57M25, 57R17; 57R58

1 Introduction

The study of Legendrian and transverse knots is central in contact geometry. A Legendrian knot with a given knot type has two classical invariants: its Thurston-Bennequin number and its rotation number. The problem of classifying Legendrian knots up to Legendrian isotopy naturally leads to the question whether these invariants classify Legendrian knots. A knot type is called Legendrian simple if any two realizations of it with equal classical invariants are Legendrian isotopic. For transverse knots there is only one classical invariant, the self-linking number. Similarly, the knot types for which transverse realizations are classified by the self-linking number are called transversely simple. The unknot [4], torus knots and the figure-eight knot [6] were proved to be both Legendrian and transversely simple. By constructing a new invariant for Legendrian knots, Chekanov [3] showed that not all knots are Legendrian simple, in particular he proved that the knot 5_2 is not Legendrian simple. Later many other Legendrian non-simple knots were detected by Epstein, Fuchs and Meyer [5], and by Ng [13]. The case for transverse knots turned out to be harder. Birman and Menasco [1], and Etnyre and Honda [9] constructed families of transversely non-simple knots using braid and convex surface theory. Recently Ng, Ozsváth and Thurston [14] gave such examples using the Legendrian invariant in knot Floer homology.

Heegaard Floer homology $\widehat{HF}(Y)$, $HF^-(Y)$ defined by Ozsváth and Szabó [17] are invariants for three-manifolds. The construction was extended [16] to give the invariants $\widehat{\mathrm{HFK}}(Y,K)$, $\mathrm{HFK}^-(Y,K)$ for null-homologous knots $K \subset Y$ via doubly pointed Heegaard diagrams. Using Heegaard diagrams with multiple basepoints the invariants were generalized for links [15]. Multiply pointed Heegaard diagrams turned out to be extremely useful in the case of knots as well, and led to the discovery of a combinatorial

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version of knot Floer homologies through grid diagrams [12, 11]. This version provided a natural way to define invariants λ_+ and λ_- of Legendrian and θ for transverse knots in the three-sphere [18].

From hereon if not stated otherwise every (Legendrian or transverse) knot is oriented and will be considered in the standard contact three-sphere. Let $m(K)$ denote the mirror of a knot. In this paper we will prove

Theorem 1.1 *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2. Then there is an isomorphism*

$$\mathrm{HFK}^-(m(K_1)) \otimes_{\mathbb{Z}_2[U]} \mathrm{HFK}^-(m(K_2)) \to \mathrm{HFK}^-(m(K_1 \# K_2))$$

which maps $\lambda_+(L_1) \otimes \lambda_+(L_2)$ to $\lambda_+(L_1 \# L_2)$. Similar statement holds for the λ_--invariant.

Corollary 1.2 *Let L_1 and L_2 be (oriented) Legendrian knots of topological type K_1 and K_2. Then there is an isomorphism*

$$\widehat{\mathrm{HFK}}(m(K_1)) \otimes_{\mathbb{Z}_2} \widehat{\mathrm{HFK}}(m(K_2)) \to \widehat{\mathrm{HFK}}(m(K_1 \# K_2))$$

which maps $\widehat{\lambda}_+(L_1) \otimes \widehat{\lambda}_+(L_2)$ to $\widehat{\lambda}_+(L_1 \# L_2)$. Similar statement holds for the $\widehat{\lambda}_-$-invariant. □

Similar results hold for the θ-invariant of transverse knots:

Corollary 1.3 *Let T_1 and T_2 be transverse knots of topological type K_1 and K_2. Then there are isomorphisms*

$$\mathrm{HFK}^-(m(K_1)) \otimes_{\mathbb{Z}_2[U]} \mathrm{HFK}^-(m(K_2)) \to \mathrm{HFK}^-(m(K_1 \# K_2))$$

and

$$\widehat{\mathrm{HFK}}(m(K_1)) \otimes_{\mathbb{Z}_2} \widehat{\mathrm{HFK}}(m(K_2)) \to \widehat{\mathrm{HFK}}(m(K_1 \# K_2))$$

which map $\theta(T_1) \otimes \theta(T_2)$ to $\theta(T_1 \# T_2)$ and $\widehat{\theta}(T_1) \otimes \widehat{\theta}(T_2)$ to $\widehat{\theta}(T_1 \# T_2)$, respectively. □

As an application of the above result we prove:

Theorem 1.4 *There exist infinitely many transversely non-simple knots.*

Similar statement follows from the main result of [8], see also [10] and [1].

The paper is organized as follows. In Section 2 we recall the definitions and collect the basic facts about Legendrian and transverse knots, knot Floer homology, and the

Legendrian and transverse invariant. In Section 3 we introduce spherical grid diagrams and prove Theorem 1.1. In Section 4 we use the results of Section 3 to prove Theorem 1.4.

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2 Preliminaries

2.1 Legendrian and transverse knots

A *Legendrian knot* L in $\mathbb{R}^3$ (or in $S^3 = \mathbb{R}^3 \cup \{\infty\}$) endowed with the standard contact form $dz - ydx$ is an oriented knot along which the form $dz - ydx$ vanishes identically. Legendrian knots are determined by their front projection to the xz-plane; a generic projection is smooth in all but finitely many cusp points, has no vertical tangents and at each crossing the strand with smaller slope is in the front. Note that if (x, z) is the standard positive basis in the plane, then in order to have the standard orientation on $\mathbb{R}^3$ the y axis points into the page. By changing the parts with vertical tangents to cusps and adding zig-zags, a generic smooth projection of a knot can be arranged to be of the above type. Thus any knot can be placed in Legendrian position. But this can be done in many different ways up to Legendrian isotopy. For example, by adding extra zig-zags in the front projection we obtain a different Legendrian representative. This method is called *stabilization*. Adding a downward (upward) cusp is called positive (negative) stabilization. (Here and throughout the paper we use the conventions of [7].) There are two classical invariants for Legendrian knots defined as follows. By pushing off the knot in the $\frac{\partial}{\partial z}$ direction we obtain the *Thurston-Bennequin framing* of the Legendrian knot. Comparing this to the Seifert framing we get the *Thurston-Bennequin number* $\mathrm{tb}(L)$. The *rotation number* $\mathrm{r}(L)$ is the winding number of TL with respect to a trivialization of the contact planes along L that extends to a Seifert surface.

A *transverse knot* in S^3 with the standard contact structure is a knot along which the contact form $dz - ydx$ never vanishes. Any transverse knot is naturally endowed with an orientation, the one along which the contact form is positive. Again, every knot can be placed in transverse position by translating its Legendrian realization in the

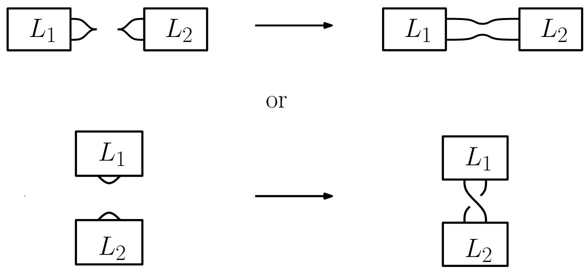


Figure 1: Connected sum of two Legendrian knots

$\pm\frac{\partial}{\partial y}$ direction. The resulting transverse knot is called the *transverse push off* of the Legendrian knot. A push off is called positive if the orientation of the knot agrees with the natural orientation of the transverse knot and called negative otherwise. A Legendrian knot is a *Legendrian approximation* of its positive push off. Two transverse knots are transversely isotopic if and only if their Legendrian approximations have common negative stabilizations. By pushing off the transverse knot T in a direction of a vector field in the contact planes that extends to a nonzero vectorfield to a Seifert-surface of T we get T'. The *self-linking number* $\mathrm{sl}(T)$ is the linking of T with its push-off T'. The self-linking number of a push off can be deduced from the classical invariants of the Legendrian knot: $\mathrm{sl}(L_\pm) = \mathrm{tb}(L) \mp \mathrm{r}(L)$.

A knot is called *Legendrian simple* (or *transversely simple*) if any two Legendrian (transverse) realizations of it with equal Thurston-Bennequin and rotation (self-linking) number(s) are isotopic through Legendrian (transverse) knots.

As it is explained in [8], there is a well-defined notion of the connected sum of two Legendrian or transverse knots in S^3, which comes from connected summing the two S^3's the knots are sitting in. This process can be described in terms of the front projection as it is shown by Figure 1.

2.2 Knot Floer homology with multiple basepoints

Here we outline the basic definitions of knot Floer homologies with multiple basepoints, originally defined by Ozsváth and Szabó [15] and independently by Rasmussen [19]. Consider a knot K in an oriented, closed three-manifold Y. There is a self-indexing Morse function with k minima and k maxima such that K is made out of $2k$ flow lines connecting all the index zero and index three critical points. Such a Morse function gives rise to a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for (Y, K) in the following way. Let $\Sigma = f^{-1}(\frac{3}{2})$ be a genus g surface. The α-curves $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{g+k-1}$ are defined to be the circles of Σ whose points flow down to the index one critical points. Similarly $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^{g+k-1}$ are the curves with points flowing up to the index two critical points. Finally let $\mathbf{w} = \{w_i\}_{i=1}^{k}$ be the positive and $\mathbf{z} = \{z_i\}_{i=1}^{k}$ the negative intersection points of K with Σ.

Consider the module $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ over the polynomial algebra $\mathbb{Z}_2[U_1, \ldots, U_k]$ freely generated by the intersection points of the totally real submanifolds $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k-1}$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k-1}$ of $\mathrm{Sym}^{g+k-1}(\Sigma)$. This module is endowed with the differential

$$\partial^- \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \\ \mu(\phi)=1}} \left| \frac{\mathcal{M}(\phi)}{\mathbb{R}} \right| U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \mathbf{y}$$

where, as usual, $\pi_2(\mathbf{x}, \mathbf{y})$ denotes the space of homotopy classes of Whitney disks connecting $\mathbf{x}$ to $\mathbf{y}$, $\mathcal{M}(\phi)$ denotes the moduli space of pseudo-holomorphic representatives of ϕ, the Maslov index $\mu(\phi)$ denotes its formal dimension and $n_p(\phi) = \#\{\phi^{-1}(p \times \mathrm{Sym}^{g+k-2}(\Sigma))\}$ is the local multiplicity of ϕ at p. Let

$$(1) \qquad \left(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}), \widehat{\partial}\right) = \left(\frac{CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})}{(U_1 = 0)}, \left[\partial^-\right] \right).$$

The chain-homotopy type of the above complexes are invariants of Y in the following sense:

Theorem 2.1 *(Ozsváth-Szabó, [15]) Let Y be a closed oriented three-manifold. Consider the Heegaard diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2)$ for Y with $|\mathbf{w}_1| = k_1$ and $|\mathbf{w}_2| = k_2$. Assuming $k_1 \geq k_2$ the complexes $CF^-(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1)$ and $CF^-(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2)$ are chain-homotopy equivalent as $\mathbb{Z}_2[U_1, \ldots, U_{k_1}]$-modules. Here the latter complex is endowed with the $\mathbb{Z}_2[U_1, \ldots, U_{k_1}]$-module structure by defining the action of $U_{k_2}, \ldots, U_{k_1}$ to be identical. Similar statement holds for the $\widehat{CF}$-theory, moreover the chain-homotopy equivalences form a commutative diagram with the factorization map of (1).* □

Hereafter we assume that our underlying three-manifold is the three-sphere. Note that in this case the homology of $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w})$ is $HF^-(S^3) = \mathbb{Z}_2[U]$. The relative Maslov-grading of two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is defined by $M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2\sum n_{w_i}(\phi)$, where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is any homotopy class from $\mathbf{x}$ to $\mathbf{y}$. We extend this relative grading to the whole module by $M(U_1^{a_1} \cdots U_k^{a_k}\mathbf{x}) = M(\mathbf{x}) - 2(a_1 + \cdots + a_k)$. For S^3, the grading can be lifted to an absolute grading by fixing the grading of the generator of $HF^-(S^3) = \mathbb{Z}_2[U]$ at 0.

Note that so far we made no reference to the basepoints $\mathbf{z}$. The relative Alexander grading is defined by $A(\mathbf{x})-A(\mathbf{y})=\sum n_{z_i}(\phi)-\sum n_{w_i}(\phi)$, where again ϕ can be chosen to be any homotopy class in $\pi_2(\mathbf{x},\mathbf{y})$. This relative grading can be uniquely lifted to an absolute Alexander grading which satisfies $\sum T^{A(x)}=\Delta_K(T)(1-T)^{n-1} \pmod 2$, where $\Delta_K(T)$ is the symmetrized Alexander polynomial. We can extend the Alexander grading to the module by $A(U_1^{a_1}\cdots U_k^{a_k}\mathbf{x})=A(\mathbf{x})-(a_1+\cdots+a_k)$. As the local multiplicities of pseudo-holomorhic discs are non-negative, we obtain filtered chain complexes $CFK^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ and $\widehat{CFK}(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$, that are invariants of the knot:

Theorem 2.2 *(Ozsváth–Szabó, [15]) Let K be an oriented knot. Consider the Heegaard diagrams $(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\mathbf{w}_1,\mathbf{z}_1)$ and $(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\mathbf{w}_2,\mathbf{z}_2)$ for K with $|\mathbf{w}_1|=|\mathbf{z}_1|=k_1$ and $|\mathbf{w}_2|=|\mathbf{z}_2|=k_2$. Assuming $k_1\geq k_2$ the filtered complexes $CFK^-(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\mathbf{w}_1,\mathbf{z}_1)$ and $CFK^-(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\mathbf{w}_2,\mathbf{z}_2)$ are filtered chain-homotopy equivalent as $\mathbb{Z}_2[U_1,\dots,U_{k_1}]$-modules. Here the latter complex is endowed with the $\mathbb{Z}_2[U_1,\dots,U_{k_1}]$-module structure by defining the action of $U_{k_2},\dots,U_{k_1}$ to be identical. Similar statement holds for the $\widehat{CFK}$-theory, moreover the chain homotopy equivalences form a commutative diagram with the factorization map of (1).* □

As it is easier to work with, we usually consider the associated graded objects of the filtered chain complexes and denote their homologies by HFK^-. In particular $HFK^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ is the homology of the complex $(CFK^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z}),\partial_0^-)$, where

$$\partial_0^-\mathbf{x}=\sum_{\mathbf{y}\in\mathbb{T}_\alpha\cap\mathbb{T}_\beta}\ \sum_{\substack{\phi\in\pi_2(\mathbf{x},\mathbf{y})\\ n_{z_1}(\phi)+\cdots+n_{z_k}(\phi)=0\\ \mu(\phi)=1}}\left|\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right|U_1^{n_{w_1}(\phi)}\cdots U_k^{n_{w_k}(\phi)}\mathbf{y}.$$

The U_i's for different i act chain-homotopically, so on the homology level all U_i act identically. This observation endows $HFK^-(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\mathbf{w},\mathbf{z})$ with a $\mathbb{Z}_2[U]$-structure, by defining the U action to be the the action of any of the U_i's. Then Theorem 2.2 translates:

Theorem 2.3 *(Oszváth–Szabó, [15]) Let K be an oriented knot. Consider the Heegaard diagrams $(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\mathbf{w}_1,\mathbf{z}_1)$ and $(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\mathbf{w}_2,\mathbf{z}_2)$ for K. Then the knot Floer homologies $\mathrm{HFK}^-(\Sigma_1,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1,\mathbf{w}_1,\mathbf{z}_1)$ and $\mathrm{HFK}^-(\Sigma_2,\boldsymbol{\alpha}_2,\boldsymbol{\beta}_2,\mathbf{w}_2,\mathbf{z}_2)$ are isomorphic as $\mathbb{Z}_2[U]$-modules. Similar statement holds for the $\widehat{\mathrm{HFK}}$-theory, moreover the isomorphisms form a commutative diagram with the factorization map of (1).* □

Knot Floer homology satisfies a Künneth-type formula for connected sums:

Theorem 2.4 *(Ozsváth–Szabó, [15]) Let K_1 and K_2 be oriented knots in S^3 described by the Heegaard diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2)$. Let $w \in \mathbf{w}_1$ and $z \in \mathbf{z}_2$. Then*

(1) $\big(\Sigma_1\#\Sigma_2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, (\mathbf{w}_1 - w) \cup \mathbf{w}_2, \mathbf{z}_1 \cup (\mathbf{z}_2 - z)\big)$ *is a Heegaard diagram for $K_1\#K_2$. Here the connected sum $\Sigma_1\#\Sigma_2$ is taken in the regions containing $w \in \Sigma_1$ and $z \in \Sigma_2$;*

Let $|\mathbf{w}_1| = |\mathbf{z}_1| = k_1$ and $|\mathbf{w}_2| = |\mathbf{z}_2| = k_2$. Both complexes $CFK^-(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1)$ and $CFK^-(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2)$ are $\mathbb{Z}_2[U_1, \dots, U_{k_1}, V_1, \dots, V_{k_2}]$-modules with the elements $U_1, \dots, U_{k_1}$ acting trivially on the latter and $V_1, \dots, V_{k_2}$ acting trivially on the former complex. With these conventions in place we have

(2) $CFK^-\left(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1\right) \otimes_{U_1=V_1} CFK^-\left(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2\right)$ *is filtered chain homotopy equivalent to*

$$CFK^-\left(\Sigma_1\#\Sigma_2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, (\mathbf{w}_1 - w) \cup \mathbf{w}_2, \mathbf{z}_1 \cup (\mathbf{z}_2 - z)\right);$$

(3) $\mathrm{HFK}^-(K_1\#K_2)$ *is isomorphic to* $\mathrm{HFK}^-(K_1) \otimes \mathrm{HFK}^-(K_2)$ *and this isomorphism can be given by* $\mathbf{x}_1 \otimes \mathbf{x}_2 \mapsto (\mathbf{x}_1, \mathbf{x}_2)$ *on the generators.*

Similar statement holds for the $\widehat{CFK}$-theory, moreover the chain homotopy equivalences form a commutative diagram with the factorization map of (1). □

2.3 Grid diagrams

As it was observed in [12, 11], knot Floer homology admits a completely combinatorial description via grid diagrams. A *grid diagram* G is a $k \times k$ square grid placed on the plane with some of its cells decorated with an X or an O and containing exactly one X and O in each of its rows and columns. Such a diagram naturally defines an oriented link projection by connecting the O's to the X's in each row and the X's to the O's in the columns and letting the vertical line to overpass at the intersection points. For simplicity we will assume that the corresponding link is a knot K. There are certain moves of the grid diagram that do not change the (topological) knot type [18]. These are *cyclic permutation* of the rows or columns, *commutation* of two consecutive rows (columns) such that the X and the O from one row (column) does not separate the X and the O from the other row (column) and *(de)stabilization* which is defined as follows. A square in the grid containing an X (O) can be subdivided into four squares by introducing a new vertical and a new horizontal line dividing the row and the column that contains that square. By replacing the X (O) by one O (X) and two X's (O's) in the diagonal of the new four squares and placing the two O's (X's) in the

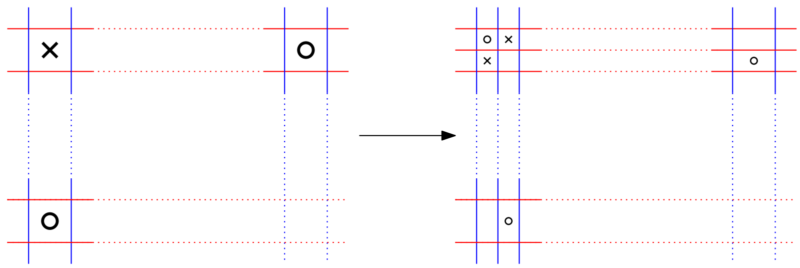


Figure 2: Stabilization of type $X\!:\!NW$

subdivided row and column appropriately, we get a new grid diagram which is called the stabilization of the original one. The inverse of stabilization is destabilization. There are eight types of (de)stabilization: $O:SW$, $O:SE$, $O:NW$, $O:NE$, $X:SW$, $X:SE$, $X:NW$ and $X:NE$, where the first coordinate indicates which symbol we started with and the second shows the placement of the unique new symbol. A stabilization of type $X:NW$ is depicted on Figure 2.

Placing the grid on a torus by identifying the opposite edges of the square grid we obtain a Heegaard diagram with multiple basepoints for (S^3, K). Here the $\mathbf{w}$'s correspond to the O's, the $\mathbf{z}$'s to the X's, and the α-curves to the horizontal lines and the β-curves to the vertical lines. As each region of this Heegaard diagram is a square, it is“nice” in the sense defined in [20]. Thus boundary maps can be given by rectangles. This observation led to a completely combinatorial description of knot Floer homology [12, 11] in the three-sphere.

2.4 Legendrian and transverse invariants on grid diagrams

Consider a grid diagram G. It describes not only a knot projection but also a front projection of a Legendrian realization of its mirror $m(K)$, as follows. Rotate the grid diagram by 45° clockwise, reverse the over- and under crossings and turn the corners into cusps or smooth them as appropriate. Legendrian Reidemeister moves correspond to certain grid moves giving the following result:

Proposition 2.5 *(Ozsváth–Szabó–Thurston, [18]) Two grid diagrams represent the same Legendrian knot if and only if they can be connected by a sequence of cyclic permutation, commutation, and (de)stabilization of types $X:NW$, $X:SE$, $O:NW$ and $O:SE$.* □

Moreover stabilizations of type $X:NE$ or $O:SW$ of the grid diagram correspond to negative stabilization of the knot, yielding

Proposition 2.6 *(Ozsváth–Szabó–Thurston, [18]) Two grid diagram represent the same transverse knot if and only if they can be connected by a sequence of cyclic permutation, commutation, and (de)stabilization of types $X:NW$, $X:SE$, $X:NE$, $O:NW$, $O:SE$ and $O:SW$.* □

Consider a grid diagram G for a Legendrian knot L of knot type K. Pick the upper right corner of every cell containing an X. This gives a generator of $CFK^-(m(K))$ denoted by $\mathbf{x}_+(G)$. Since there is no positive rectangle starting at $\mathbf{x}_+(G)$, it is a cycle defining an element $\lambda_+(G)$ in $\mathrm{HFK}^-(m(K))$. Similarly one can define $\mathbf{x}_-(G)$ and $\lambda_-(G)$ by taking the lower left corners of the cells containing X's. These elements are proved to be independent of the grid moves that preserve the Legendrian knot type, giving an invariant of the Legendrian knot L:

Theorem 2.7 *(Ozsváth–Szabó–Thurston, [18]) Consider two grid diagrams G_1 and G_2 defining the same oriented Legendrian knot. Then there is a quasi-isomorphism of the graded chain complexes CFK^- taking $\mathbf{x}_+(G_1)$ to $\mathbf{x}_+(G_2)$ and $\mathbf{x}_-(G_1)$ to $\mathbf{x}_-(G_2)$.* □

One can understand the transformation of $\mathbf{x}_+(G)$ and $\mathbf{x}_-(G)$ under positive and negative stabilization:

Theorem 2.8 *(Ozsváth–Szabó–Thurston, [18]) Let L be an oriented Legendrian knot, denote by L_+ its positive and by L_- its negative stabilization. Then*

(1) *There is a quasi-isomorphism of the corresponding graded complexes taking $\mathbf{x}_+(L)$ to $\mathbf{x}_+(L_+)$ and $U\mathbf{x}_-(L)$ to $\mathbf{x}_-(L_+)$;*

(2) *There is a quasi-isomorphism of the corresponding graded complexes taking $U\mathbf{x}_+(L)$ to $\mathbf{x}_+(L_-)$ and $\mathbf{x}_-(L)$ to $\mathbf{x}_-(L_-)$.*

□

It follows from [7] that the Legendrian knots with transversely isotopic positive push offs admit common negative stabilizations. This principle provides a well defined invariant for transverse knots: if L is a Legendrian approximation of T then define $\theta(T) = \lambda_+(L)$.

Theorem 2.9 *(Ozsváth–Szabó–Thurston, [18]) For any two grid diagrams G_1 and G_2 of Legendrian approximations of the transverse knot T there is quasi-isomorphism of the corresponding graded chain complexes inducing a map on the homologies that takes $\theta(G_1)$ to $\theta(G_2)$.* □

3 Proof of Theorem 1.1

The Legendrian invariant can be thought of in two different ways, depending on the version of Floer homology we work with. The one introduced in subsection 2.4 is in the combinatorial Heegaard Floer homology. Once the grid is placed on the torus we get a Heegaard diagram and thus there is a natural identification of the combinatorial Heegaard Floer complex with the holomorphic Heegaard Floer complex [11]. Under this identification the previously defined invariant has a counterpart in the original, holomorphic Heegaard Floer homology. We will use the same notation for both. In the next subsection we introduce yet another invariant for Legendrian knots.

3.1 Legendrian invariant on spherical Heegaard diagrams

A $k \times k$ grid diagram G of a Legendrian knot L of topological type K can also be placed on the 2-sphere in the following way. Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 : \ |(x,y,z)| = 1\}$ and define the circles $\widetilde{\boldsymbol{\alpha}} = \{\widetilde{\alpha}_i\}_{i=1}^{k-1}$ as the intersection of S^2 with the planes $A_i = \{(x,y,z) \in \mathbb{R}^3 : \ z = \frac{i}{k} - \frac{1}{2}\}$ $(i = 1, \ldots, k-1)$; similarly define $\widetilde{\boldsymbol{\beta}} = \{\widetilde{\beta}_i\}_{i=1}^{k-1}$ as the intersection of S^2 with the planes $B_i = \{(x,y,z) \in \mathbb{R}^3 : \ x = \frac{i}{k} - \frac{1}{2}\}$ $(i = 1, \ldots, k-1)$. Call $F = \{(x,y,z) \in \mathbb{R}^3 : \ |(x,y,z)| = 1, y \geq 0\}$ the front hemisphere, and $R = \{(x,y,z) \in \mathbb{R}^3 : \ |(x,y,z)| = 1, y \leq 0\}$ the rear hemisphere. Then there is a grid on both the front and on the rear hemisphere. We place the X's and the O's on the front hemisphere in the way they were placed on the original grid G. After identifying the O's with $\widetilde{\mathbf{w}} = \{\widetilde{w}_i\}_{i=1}^{k}$ and the X's with $\widetilde{\mathbf{z}} = \{\widetilde{z}_i\}_{i=1}^{k}$ this defines a Heegaard diagram $(S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ with multiple basepoints for (S^3, K). A spherical grid diagram for the trefoil knot is shown by Figure 4.

Let L be a Legendrian knot in S^3. To define the spherical Legendrian invariant $\lambda_+^S(L)$ we will use a grid diagram that have an X in its upper right corner. This can always be arranged by cyclic permutation, but in the following we will need a slightly stronger property:

Lemma 3.1 *For any Legendrian knot there exists a grid diagram representing it which contains an X in its upper right corner and an O in its lower left corner.*

Proof Consider any grid diagram describing the Legendrian knot L. As it is illustrated on Figure 3, we can obtain a suitable diagram as follows. First do a stabilization of type $X{:}NE$ and then do a stabilization of type $O{:}NE$ on the newly obtained O. Lastly, by cyclic permutation we can place the lower X introduced in the first stabilization to the upper right corner of the diagram. Notice that the O on the upper right of this X will be automatically placed to the lower left corner. According to Proposition 2.5 the Legendrian type of the knot is fixed under these moves, thus the statement follows.

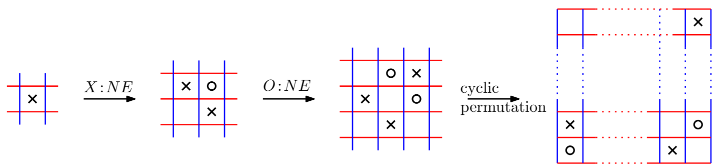


Figure 3: Grid moves

□

Suppose, that G is a grid diagram having an X in its upper right corner. Form a spherical grid diagram as above. Define $\mathbf{x}_+^S(L)$ as the generator of $CFK^-(S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ consisting of those intersection points on the front hemisphere that occupy the upper right corner of each region marked with an X. Note that the X in the upper right corner has no such corner. On Figure 4 the element $\mathbf{x}_+^S$ is indicated for the trefoil knot. Similarly to the toroidal case we have:

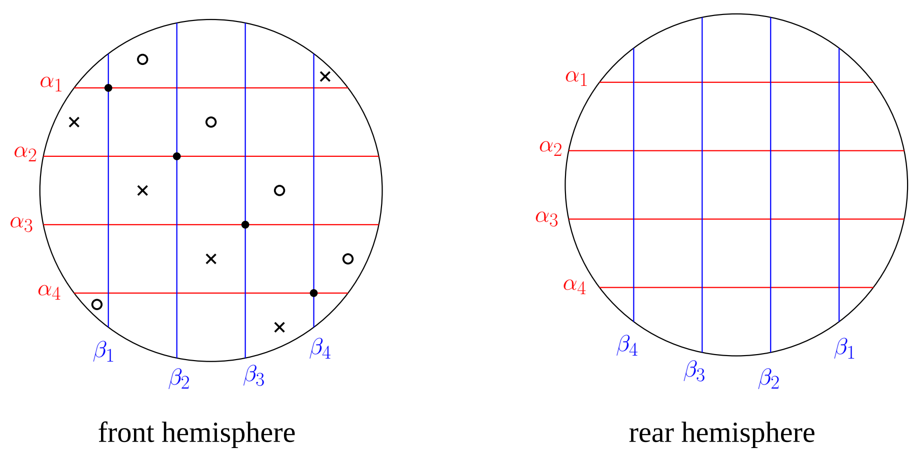


Figure 4: Spherical grid diagram for the trefoil knot

Lemma 3.2 *The element $\mathbf{x}_+^S(L)$ is a cycle in $(S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$.*

Proof We will show that for any $\mathbf{y}$ there is no positive disc $\psi \in \pi_2(\mathbf{x}_+^S, \mathbf{y})$ with $\mu(\psi) = 1$. As the diagram $CFK^-(S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ is "nice" in the sense of [20] the elements $\mathbf{x}_+^S$ and $\mathbf{y}$ differ either in one coordinate and $\mathcal{D}(\psi)$ is a bigon or they differ in two coordinates and $\mathcal{D}(\psi)$ is a rectangle. In any case, $\mathcal{D}(\psi)$ contains an X which means it is not counted in the boundary map. □

The homology class of $\mathbf{x}_+^S$, denoted by $\lambda_+^S(G)$, turns out to be an invariant of L (i.e. it is independent of the choice of the grid diagram, and the way it is placed on the sphere). This can be proved directly through grid moves, but instead we show:

Theorem 3.3 *Consider a grid diagram for the Legendrian knot L in S^3 having an X in its upper right corner. Then there is a filtered quasi-isomorphism ψ : $CFK^-(T^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z}) \to CFK^-(S^2, \widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{\beta}}, \widetilde{\mathbf{w}}, \widetilde{\mathbf{z}})$ of the corresponding toroidal and spherical Heegaard diagrams which maps $\mathbf{x}_+(L)$ to $\mathbf{x}_+^S(L)$.*

In the proof we will need the notion of Heegaard triples, which we will briefly describe here. (For a complete discussion see [17].) Consider a pointed Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$. The pairs $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z})$, $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$ define the three-manifolds $Y_{\alpha\beta}$, $Y_{\gamma\beta}$ and $Y_{\alpha\gamma}$, respectively. There is a map from $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{z}) \otimes CF^-(\Sigma, \boldsymbol{\beta}, \boldsymbol{\gamma}, \mathbf{z})$ to $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\gamma}, \mathbf{z})$ given on a generator $\mathbf{x} \otimes \mathbf{y}$ by

$$\sum_{\mathbf{v} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{u \in \pi_2(\mathbf{x},\mathbf{y},\mathbf{v}) \\ n_{\mathbf{z}}(u)=0 \\ \mu(u)=0}} |\mathcal{M}(u)| \mathbf{v}$$

where $\pi_2(\mathbf{x}, \mathbf{y}, \mathbf{v})$ is the set of homotopy classes of triangles connecting $\mathbf{x}$, $\mathbf{y}$ to $\mathbf{v}$; maps from a triangle to $\mathrm{Sym}^{g+k-1}(\Sigma)$ sending the edges of the triangle to $\mathbb{T}_\alpha, \mathbb{T}_\beta$ and $\mathbb{T}_\gamma$, $\mathcal{M}(u)$ is the moduli space of pseudo-holomorphic representatives of the homotopy class u. This gives a well-defined map on the homologies HF^-. When $\boldsymbol{\gamma}$ can be obtained from $\boldsymbol{\beta}$ by Heegaard moves then the manifold $Y_{\beta\gamma}$ is $\#^g S^1 \times S^2$ and $HF^-(\#^g S^1 \times S^2)$ is a free $\mathbb{Z}_2[U]$-module generated by 2^g-elements. Denote its top-generator by $\Theta^-_{\beta\gamma}$. The same definition gives a map on the filtered chain complexes CFK^-. The map $CFK^-(Y_{\alpha\beta}) \to CFK^-(Y_{\alpha\gamma})$ sending $\mathbf{x}$ to the image of $\mathbf{x} \otimes \Theta^-_{\beta\gamma}$ defines a quasi-isomorphism of the chain complexes.

Proof of theorem 3.3 From a toroidal grid diagram one can obtain a spherical one by first sliding every β-curve over β_1 to obtain $\boldsymbol{\beta}'$ and sliding every α-curve over α_1 to obtain $\boldsymbol{\alpha}'$, and then destabilize the diagram at α_1 and β_1. Thus we will construct the quasi-isomorphism by the composition $\psi = \psi_{\mathrm{destab}} \circ \psi_\alpha \circ \psi_\beta$, where

$$\psi_\beta = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_{\beta'}} \sum_{\substack{u \in \pi_2(\mathbf{x}_+(L), \Theta^-, \mathbf{y}) \\ n_{\mathbf{z}}(u)=0 \\ \mu(u)=0}} |\mathcal{M}(u)| \mathbf{y}$$

with $\Theta^- \in \mathbb{T}_\beta \cap \mathbb{T}_{\beta'}$ being the top generator of $HF^-(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z}) = HF^-(S^1 \times S^2)$ and ψ_α defined similarly. Note that in the case of the sliding there is also a "closest point" map denoted by $'$ for the sliding of the β-curves and by $''$ for the sliding of the α-curves. We claim:

Lemma 3.4 $\psi_\beta(\mathbf{x}_+) = \mathbf{x}'_+$.

Lemma 3.5 $\psi_\alpha(\mathbf{x}'_+) = (\mathbf{x}'_+)''$.

Here we just include the proof of Lemma 3.4; Lemma 3.5 follows similarly.

Proof of Lemma 3.4 Figure 5 shows a weakly admissible diagram for the slides of the β-curves.

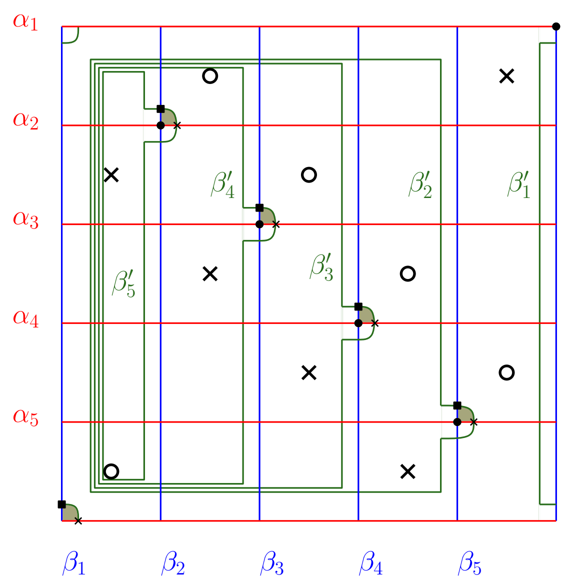


Figure 5: Handleslides

Claim 1 *The Heegaard triple* $(T^2, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$ *of Figure 5 is weakly admissible.*

Proof Let $\mathcal{P}_{\beta_i\beta'_i\beta_1}$ $(i > 1)$ denote the domain bounded by β_i, β'_i and β_1 and containing no basepoint. Similarly $\mathcal{P}_{\beta_1\beta'_1}$ denotes the domain bounded by β_1 and β'_1 and containing no basepoint. These domains form a basis for the periodic domains of $(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$ and as all have domains with both positive and negative coefficients we can see that $(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$ is weakly admissible. Consider a triply periodic domain $\mathcal{P}$. If there is no α-curve in its boundary, then it is a periodic domain of $(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$, and by the previous observation we are done. So $\mathcal{P}$ must contain an α-curve in its boundary. To ensure it does not contain an X, there must be some vertical curve, either from β or β', in the boundary. At the intersection point of the horizontal and vertical lines the domain must change sign, concluding the argument. $\square$

The grey area in Figure 5 indicates a domain of a canonical triangle u_0 connecting $\mathbf{x}_+(L), \Theta^-$ and $\mathbf{x}'_+(L)$; by the Riemann mapping theorem there is exactly one map with that domain. We claim that this is the only map that is encountered in ψ_β. For this, let $u \in \pi_2(\mathbf{x}_+(L), \Theta^-, \mathbf{y})$ be a holomorphic triangle with $\mu(u) = 0$ and $n_{\mathbf{z}}(u) = 0$.

Claim 2 *There exists a periodic domain $\mathcal{P}_{\beta\beta'}$ of $(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$ such that $\partial(\mathcal{D}(u) - \mathcal{D}(u_0) - \mathcal{P}_{\beta\beta'})|_{\boldsymbol{\beta}} = \emptyset$. Thus $(\mathcal{D}(u) - \mathcal{D}(u_0) - \mathcal{P}_{\beta\beta'})|_{\boldsymbol{\beta}}$ is a domain in $(T^2, \boldsymbol{\alpha}, \boldsymbol{\beta'}, \mathbf{z})$, representing an element v in $\pi_2(\mathbf{x}'_+, \mathbf{y})$ with Maslov index $\mu(v) = \mu(u) - \mu(u_0) - \mu(\mathcal{P}_{\beta\beta'}) = 0$.*

Proof As $n_{\mathbf{z}}(u) = 0$ and $\mathbf{x}'_+(L)$ is in the upper right corner of the X's, the domain of any triangle must contain $\mathcal{D}(u_0)$. Consequently $\partial\mathcal{D}(u)|_{\beta_i}$ is an arc containing the small part $\overline{\mathcal{D}(u_0)} \cap \beta_i$ and some copies of the whole β_i. By subtracting $\mathcal{D}(u_0)$ and sufficiently many copies of the periodic domains $\mathcal{P}_{\beta_i\beta'_i\beta_1}$ we obtain a domain with no boundary component on β_i. Doing the same process for every $i > 1$ and then by subtracting some $\mathcal{P}_{\beta_1\beta'_1}$ we can eliminate every β_i from the boundary. $\square$

Claim 3 *There is no positive disc in $\pi_2(\mathbf{x}'_+, \mathbf{y})$.*

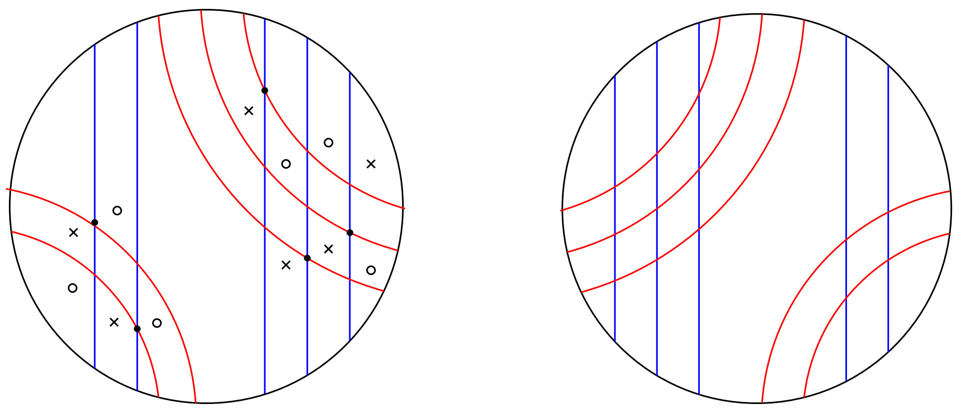
Proof This follows similarly to Lemma 3.2. $\square$

Claim 4 *None of the regions of $(T^2, \boldsymbol{\alpha}, \boldsymbol{\beta'}, \mathbf{z})$ can be covered completely with the periodic domains of $(T^2, \boldsymbol{\beta}, \boldsymbol{\beta'}, \mathbf{z})$ and $\mathcal{D}(u_0)$.*

Proof The periodic domains are the linear combinations of $\{\mathcal{P}_{\beta_i,\beta'_i,\beta_1}\}_{i=2}^k \cup \{\mathcal{P}_{\beta_1,\beta'_1},\}$, and those cannot cover the domains of $(T^2, \boldsymbol{\alpha}, \boldsymbol{\beta'}, \mathbf{z})$. $\square$

Putting these together, we have that $\mathcal{D}(u) - \mathcal{D}(u_0) - \mathcal{P}_{\beta\beta'}$ has a negative coefficient, which gives a negative coefficient in $\mathcal{D}(u)$ as well, contradicting the fact that u was holomorphic. This proves Lemma 3.4. $\square$

Note that by assuming that there is an X in the upper right corner of the grid diagram we assured that the intersection point $\mathbf{x}_+$ contains $\alpha_1 \cap \beta_1$, and that point remained unchanged during the whole process. Thus by destabilizing at α_1 and β_1 we get Theorem 3.3. $\square$



front hemisphere rear hemisphere

Figure 6: Connected Sum

Proof of Theorem 1.1 Consider two Legendrian knots L_1 and L_2 of topological types K_1 and K_2. Note that once we obtain the result for λ_+^S we are done. Indeed, passing from the toroidal diagram to the spherical one, the invariants $\lambda_+(L_1)$ and $\lambda_+(L_2)$ are mapped to $\lambda_+^S(L_1)$ and $\lambda_+^S(L_2)$, respectively. Knowing that $\lambda_+^S(L_1) \otimes \lambda_+^S(L_2)$ is mapped to $\lambda_+^S(L_1 \# L_2)$ and passing back to the toroidal diagram, there is an isomorphism that maps this to $\lambda_+(L_1 \# L_2)$. So the combination of these arguments prove Theorem 1.1.

Consider the grid diagrams G_1 and G_2 corresponding to L_1 and L_2 admitting the conditions of Lemma 3.1. These grids define the spherical grid diagrams $(S^2, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1)$ and $(S^2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2)$. Let $z \in \mathbf{z}_1$, $w \in \mathbf{w}_2$ be the basepoints corresponding to the X in the upper right corner of the first diagram and the O in the lower left corner of the second diagram. Form the connected sum of $(S^2, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1)$ and $(S^2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2)$ at the regions containing z and w to obtain a Heegaard diagram with multiple basepoints $(S^2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, \mathbf{w}_1 \cup (\mathbf{w}_2 - \{w\}), (\mathbf{z}_1 - \{z\} \cup \mathbf{z}_2))$ of $(S^3, L_1 \# L_2)$. By 2.4 the map

$$\begin{aligned}\psi_{\text{connsum}}: \quad & \mathrm{HFK}^-(S^2, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, \mathbf{w}_1, \mathbf{z}_1) \otimes \mathrm{HFK}^-(S^2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, \mathbf{w}_2, \mathbf{z}_2) \to \\ & \mathrm{HFK}^-(S^2, \boldsymbol{\alpha}_1 \cup \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \cup \boldsymbol{\beta}_2, \mathbf{w}_1 \cup (\mathbf{w}_2 - \{w\}), (\mathbf{z}_1 - \{z\}) \cup \mathbf{z}_2)\end{aligned}$$

defined on the generators as $\mathbf{x}_1 \otimes \mathbf{x}_2 \mapsto (\mathbf{x}_1, \mathbf{x}_2)$ is an isomorphism. Thus the image of $\lambda_+^S(L_1) \otimes \lambda_+^S(L_2)$ is $(\lambda_+^S(L_1), \lambda_+^S(L_2))$.

Figure 6 shows the resulting Heegaard diagram. From this diagram of the connected sum one can easily obtain a spherical grid diagram by isotoping every curve in $\boldsymbol{\alpha}_1$ to intersect the curves in $\boldsymbol{\beta}_2$ and every curve in $\boldsymbol{\alpha}_2$ to intersect the curves in $\boldsymbol{\beta}_1$ as shown on Figure 7. Indeed, the resulting diagram is a grid obtained by patching G_1 and G_2 together in the upper right X of G_1 and the lower left O of G_2 and deleting the X and O at issue. Now by connecting the X in the lower row of G_2 to the O in the upper row of G_1, and proceeding similarly in the columns we get that the grid corresponds to the front projection of $L_1 \# L_2$. Again, a quasi-isomorphism ψ_{isot} is given with the help of holomorphic triangles. A similar argument as in the proof of Lemma 3.4 shows that under the isomorphism induced by ψ_{isot} on the homologies, the element $(\lambda_+^S(L_1), \lambda_+^S(L_2))$ is mapped to $\lambda_+^S(L_1 \# L_2)$.

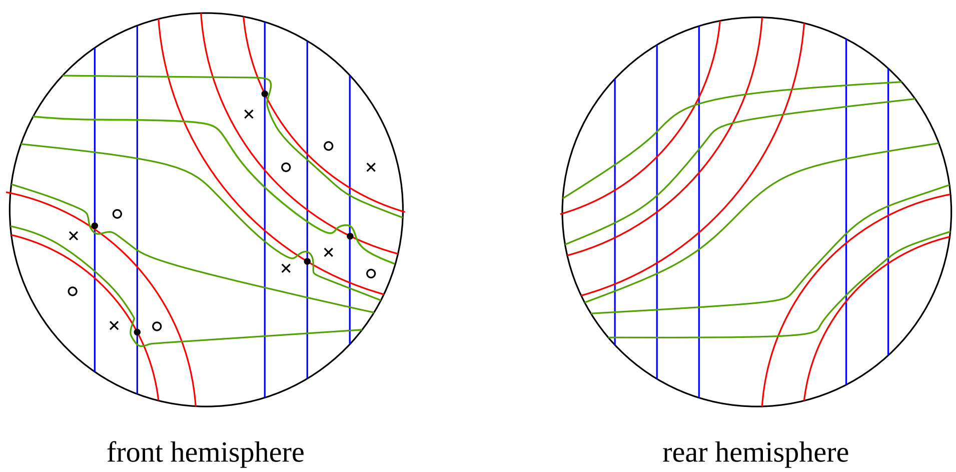


Figure 7: Isotoping to obtain a grid diagram

□

4 Proof of Theorem 1.4

One way of distinguishing transverse knots in a given knot type is to prove that their $\widehat{\theta}$-invariants are different. This, however, cannot be done straightforwardly as the vector space $\widehat{\mathrm{HFK}}$ does not canonically correspond to a knot. So in order to prove that two elements are different, we have to show that there is no isomorphism of $\widehat{\mathrm{HFK}}$ carrying one to the other. More explicitly, it is enough to see that there is no such isomorphism induced by a sequence of Heegaard moves. For instance, if we show that one element is 0, while the other is not, we can be certain that they are different. This is used in the proof of Theorem 1.4.

Proof of Theorem 1.4 Ng, Ozsváth and Thurston [14] showed that the knot type 10_{132} contains transversely non-isotopic representatives L_1 and L_2 with equal self-linking number. They proved that $\widehat{\theta}(L_1)$ is zero in $\widehat{\mathrm{HFK}}(m(10_{132}))$ while $\widehat{\theta}(L_2)$ is not. In the following we will prove that the knot types $\#^n 10_{132}$ are transversely non-simple. By the uniqueness of prime decomposition of knots [2], these are indeed different knot types. Thus this list provides infinitely many examples of transversely non-simple knots. The two transversely non isotopic representatives of $\#^n 10_{132}$ are $\#^n L_2$ and $L_1\#(\#^{n-1}L_2)$. Using the formula $\mathrm{sl}(L_1'\#L_2') = \mathrm{sl}(L_1') + \mathrm{sl}(L_2') + 1$ for the self-linking numbers we have $\mathrm{sl}(\#^n L_2) = n\mathrm{sl}(L_2) + (n-1) = \mathrm{sl}(L_1) + (n-1)\mathrm{sl}(L_2) + (n-1) = \mathrm{sl}(L_1\#(\#^{n-1}L_2))$. We use Corollary 1.2 to distinguish the transverse isotopy types of $\#^n L_2$ and $L_1\#(\#^{n-1}L_2)$. There is an isomorphism from $\widehat{\mathrm{HFK}}(m(10_{132}))\otimes\widehat{\mathrm{HFK}}(\#^{n-1}m(10_{132}))$ to $\widehat{\mathrm{HFK}}(\#^n m(10_{132}))$ mapping $\widehat{\theta}(L_1)\otimes\widehat{\theta}(\#^{n-1}L_2)) = 0$ to $\widehat{\theta}(L_1\#(\#^{n-1}L_2))$, thus it is zero. Similarly, there is an isomorphism mapping $\widehat{\theta}(L_2)\otimes\widehat{\theta}(\#^{n-1}L_2)) \neq 0$ to $\widehat{\theta}(L_2\#(\#^{n-1}L_2))$, thus by induction on n it does not vanish.

□

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