Advances in Business-Related Scientific Research Conference 2010 (ABSRC 2010)
September 8-10, 2010, Olbia, Sardinia, Italy

REVERSE OPTIMIZATION OF GROWTH OPTIMAL PORTFOLIO SELECTION

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Abstract

Our aim is to forecast stock returns using various market models. Assuming portfolio optimization policies for investors, we infer expected returns based on market behavior. We investigate three pricing models: Capital Asset Pricing Model, Carhart Four-Factor Arbitrage Pricing Model, and a Growth Optimal Pricing Model. We derive the non-parametric, multi-period growth optimal pricing formula based on growth optimal portfolio theory. Our out-of-sample tests run through Standard & Poor's 500 index's constituents from 1970 to 2008. Contrary to previous findings, our results show that the Growth Optimal Pricing Model significantly outperforms its competitors in a test of approximately 65,000 estimations.

Key Words: Asset pricing, Growth optimal investment, Reverse optimization, Return estimation

Topic Groups: Economics and business, Research methods

INTRODUCTION

In contrast to pure statistical reasoning based on efficiency of average, reverse optimization (Black & Litterman, 1992; Sharpe, 2002) uses knowledge about behavior of market participants. While investors optimize portfolios based on anticipated covariances and expected returns, we utilize the result of their optimization, the market portfolio. Estimating covariances historically, we derive expected returns that market participants have in mind. Comparing out-of-sample predictive power, our experiments establish solid ordering among the investigated models.

We compare four approaches to estimate returns: historical average, the Capital Asset Pricing Model (Sharpe, 1964; Lintner, 1965 and Mossin, 1966), the Four-Factor Arbitrage Pricing Model (Carhart, 1997) and a Growth Optimal Pricing Model. Our aim is to distinguish the model minimizing sum squared errors, therefore providing us with the most efficient estimations of expected returns. To avoid overfitting, we do out-of-sample tests on Standard

& Poor's (S&P) 500 index's constituents, using 65,000 samples. Contrary to previous research, we show significant differences in accuracy of the Growth Optimal Pricing Model (GOPM in the sequel) versus the Capital Asset Pricing Model (CAPM). Carhart's Four-Factor Model clearly underperforms, while GOPM significantly outperforms its competitors. To our knowledge this is the first test of the model in the literature by means of out-of-sample squared error.

Asset pricing models are based on assumptions about utility – return and risk preferences – of market participants. Both CAPM and GOPM estimates returns based on the stochastic relation of the given asset to the market portfolio. Both models assume rational investors with homogeneous expectations, thus the proportional portfolio of any investor coincides with the market portfolio.

CAPM assumes that investors optimize quadratic utility (see Sharpe, 2007) in a single period investment framework. The latter means they follow a buy-and-hold approach. In contrast with that, growth optimal investment of Kelly (1956) and Latané (1959) is a multi period investment strategy. Here investors consider the fact that they are allowed to rebalance their portfolios several times in the future. The strategy optimizes asymptotical average rate of growth, hence it can be shown that it outperforms any other investment strategy on the long run, almost surely. While "In the long run, we're all dead" (Keynes), GOPM has favorable properties on the short term, as well. We summarize aspects of growth optimal investment in Section Properties of Growth Optimal Investment.

After pioneering papers about empirical testing of growth optimal pricing (Roll, 1973; Fama & MacBeth, 1973), there were no other works in the literature to our knowledge. Most recent empirical studies are that of Györfi et al. (2006; 2007). Works of Roll and Fama & MacBeth investigate whether the growth optimal approach can be significantly distinguished from CAPM. Although, neither studies reject the hypothesis that New York Stock Exchange listed stocks are priced growth optimally, they do not find significant evidences in favor of GOPM versus CAPM. Fama and MacBeth conclude that while monthly returns might be characterized by any of the two models, GOPM achieves notably higher gains in sense of wealth achieved. Grauer (1981) affirms results of Fama and MacBeth on artificially generated monthly time series. In his experiments Grauer cannot significantly distinguish the two models, but observes higher gains of growth optimal investment.

While our growth optimal framework uses discrete time setting, analysis in continuous time is documented in the literature. Luenberger (1998) presents a comprehensive survey and a continuous time pricing model. Assuming lognormality of returns, he also finds that the Continuous-time Growth Optimal Pricing Formula is similar to CAPM. Also Bajeux-Besnainou and Portait (1998) analyze continuous rebalancing mean-variance efficient strategies, and find a CAPM like growth optimal pricing equation.

Difficulties in comparing the two pricing models is a consequence of their similarity. Kraus, Litzenberger (1975) and Ottucsák and Vajda (2007), show close correspondence between the mean-variance and the growth optimal approach. Our empirical tests are different from the aforementioned experiments in the sense that we improved volatility and covariance estimates by using higher frequency weekly data and exponential weighing.

INVESTMENT FRAMEWORK

Our notations for asset prices and returns are as follows. Consider a market consisting of d assets. The evolution of the market is represented by a sequence of return vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots \in R_+^d$, where

$$\mathbf{r}_t = (r_t^{(1)}, \ldots, r_t^{(d)}),$$

and $r_t^{(j)}$ denotes return of the j-th asset at the end of the t-th trading period. At any given time $\mathbf{r}_t$ is drawn randomly from the unknown probability distribution $\mathbf{R}_t$.

In our models short selling is allowed, and results with immediate income. Our only constraint is that we have to invest our total wealth:

$$\Delta_d = \left\{ \mathbf{b} = (b^{(1)}, \ldots, b^{(d)}); \langle \mathbf{b}, \mathbf{e} \rangle = 1 \right\},$$

where $\langle \cdot,\cdot \rangle$ denotes inner product, and $\mathbf{e} \in \mathrm{R}^d, \mathbf{e} = (1,1,\ldots,1)$.

Let S_0 denote the investor's initial capital. At the beginning of the first trading period $S_0 b_1^{(j)}$ is invested into asset j, and it results in wealth $S_0 b_1^{(j)}(1 + r_1^{(j)})$. At the end of the first trading period the investor's total wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b_1^{(j)}(1 + r_1^{(j)}) = S_0 \langle \mathbf{b}_1, \mathbf{e} + \mathbf{r}_1 \rangle.$$

For the second trading period S_1 is the new initial capital, hence

$$S_2 = S_1 \langle \mathbf{b}_1, \mathbf{e} + \mathbf{r}_2 \rangle = S_0 \langle \mathbf{b}_1, \mathbf{e} + \mathbf{r}_1 \rangle \langle \mathbf{b}_2, \mathbf{e} + \mathbf{r}_2 \rangle.$$

By induction, for the trading period t

$$S_t = S_{t-1} \langle \mathbf{b}_t, \mathbf{e} + \mathbf{r}_t \rangle = S_0 \prod_{i=1}^{t} \langle \mathbf{b}_i, \mathbf{e} + \mathbf{r}_i \rangle.$$

The asymptotic average growth rate of this portfolio selection is defined as:

$$\lim_{t\to\infty} \frac{1}{t} \ln S_t = \lim_{t\to\infty} \left(\frac{1}{t} \ln S_0 + \frac{1}{t} \sum_{i=1}^{t} \ln \langle \mathbf{b}_t, \mathbf{e} + \mathbf{r}_i \rangle \right)$$
$$= \lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \ln \langle \mathbf{b}_t, \mathbf{e} + \mathbf{r}_i \rangle.$$

Without loss of generality, we can assume in the sequel that the initial capital $S_0 = 1$.

Capital Asset Pricing Model

Markowitz (1952) established modern portfolio theory (MPT) assuming investors, whose utility is quadratic (see Sharpe, 2007), hence it is completely captured by mean and variance of returns. This means that for any given level of risk – i.e. variance –, investors choose the portfolio that maximizes expected return.

It is possible to show that introducing a risk-free bond with known constant interest rate, the risky part of any investor's portfolio is the same. Thus, assuming rationality of investors and homogeneous expectations, the market portfolio is mean-variance efficient, as well. Capital Asset Pricing Model (Sharpe, 1964; Lintner, 1965 and Mossin, 1966) describes the relation between expected returns, variances and covariances, that holds if and only if the market portfolio is mean-variance efficient:

$$\mathbf{E}(R_t^{(i)}) - r_t^{(f)} = \beta_t^{(i)}(\mathbf{E}(R_t^{(M)}) - r_t^{(f)}). \tag{1}$$

$R_t^{(i)}$ is the return on asset *i*, $r_t^{(f)}$ is the constant risk-free rate, and $R_t^{(M)}$ is the return of the market portfolio. The pricing equation is ofter referred to as Sharpe-Lintner-Mossin equation. From the equation if follows that

$$\beta_t^{(i)} = \frac{Cov(R_t^{(i)}, R_t^{(M)})}{Var(R_t^{(M)})}.$$

Equation (1) introduces risk premium for assets with high covariance with the market. This is similar to the Growth Optimal Pricing Model, although the latter considers higher moments, as well.

Whilst the CAPM considers a mean-variance efficient market portfolio, an efficient multi-period strategy is not mean-variance efficient between any two points in time in general (Hakansson, 1971). Being a multi-period investment framework, GOPM naturally solves this problem.

Roll (1977) criticizes the model's testability since equation (1) is true for all efficient portfolios different from the market portfolio. Testing validity of the equation is equivalent to testing mean-variance efficiency of the used market proxy. Although, mean-variance efficiency of the market portfolio is the question we pose too, our tests reveal that the model has significant predictive power, as well.

Carhart Four-Factor Model

Carhart's Four-Factor Model (Carhart, 1997) is an arbitrage pricing model based on Stephen Ross' arbitrage pricing theory (Ross, 1976). While CAPM uses a single regressor – the market portfolio –, the Four-Factor Model extends this with three other market indices: Fama and French's value-weighted indices on size, book-to-market equity ratio and Carhart's momentum portfolio. Fama and French (1993) and Carhart (1997) show that these factors significantly improved in-sample explanatory power.

The model states that

$$R_t^{(i)} - r_t^{(f)} = \alpha_t^{(i)} + \beta^{(i)}(R_t^{(M)} - r_t^{(f)}) + s^{(i)} SMB_t + h^{(i)} HML_t + mom^{(i)} MOM_t + \varepsilon_t^{(i)},$$

where $R_t^{(i)}$ denotes returns on asset *i*, $r_t^{(f)}$ is the risk-free rate and $R_t^{(M)}$ is the return of the market portfolio. Fama and French's small-minus-big factor (SMB_t) measures return difference between small and large capitalization stocks, and high-minus-low factor (HML_t) is return difference between stocks with high and low book-to-market equity ratio. The momentum factor (MOM_t) measures excess return of past winners above past losers. The regression error $\varepsilon_t^{(i)}$ has zero mean, and the intercept is $\alpha_t^{(i)}$. Assuming zero alpha, expected value of returns is expressed as

$$\mathbf{E}(R_t^{(i)}) - r_t^{(f)} = \beta^{(i)}(\mathbf{E}(R_t^{(M)}) - r_t^{(f)}) + s^{(i)}\mathbf{E}(SMB_t) + h^{(i)}\mathbf{E}(HML_t) + mom^{(i)}\mathbf{E}(MOM_t). \quad (2)$$

Roll criticizes arbitrage pricing theory based factor models (Roll, 1977). He points out that it is always possible to construct in-sample pricing models that satisfy equation (2). Hence in-sample validity of the pricing model may be the result of data dredging[1].

Growth Optimal Pricing Model

The pricing model is based on the assumption that investors follow the dynamic, multi-period, growth optimal – i.e. log-optimal – investment strategy. A representative example of dynamic portfolio selection is the constantly rebalanced portfolio (CRP), introduced and studied by Kelly (1956), Latané (1959), Breiman (1961), Markowitz (1976), Finkelstein and Whitley (1981), Móri (1984), Móri and Székely (1984) and Barron and Cover (1988). For a comprehensive survey see Cover and Thomas (1991), Luenberger (1998) and Györfi et al. (2007).

Assume independent identically distributed (i.i.d.) returns ($\mathbf{R} \in \mathbb{R}_+^d$). Following the CRP strategy we fix a proportional portfolio vector $\mathbf{b} \in \Delta_d$. Our hypothetical investor neither consumes nor deposits cash, but reinvest his portfolio at each trading period with regard to $\mathbf{b}$. Asymptotic average rate of growth of this portfolio selection is

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \ln\langle \mathbf{b}, \mathbf{e} + \mathbf{r}_t \rangle = \mathbf{E}\{\ln\langle \mathbf{b}, \mathbf{e} + \mathbf{R} \rangle\},$$

almost surely, given that $\mathbf{E}\{\ln\langle \mathbf{b}, \mathbf{e} + \mathbf{R} \rangle\}$ is finite. The key to the approach lies in the recognition that average rate of growth densifies around $\mathbf{E}\{\ln\langle \mathbf{b}, \mathbf{e} + \mathbf{R} \rangle\}$ due to Law of Large Numbers (Györfi et al., 2007). Since in general

$$\mathbf{E}\{\ln\langle \mathbf{b}, \mathbf{e} + \mathbf{R} \rangle\} \neq \mathbf{E}\{\langle \mathbf{b}, \mathbf{e} + \mathbf{R} \rangle\},$$

[1] Overfitting on past data which is not valid on other datasets.

optimization of expected return results in a misleading estimation, that is almost surely not achieved asymptotically.

The best constantly rebalanced portfolio (BCRP) is the CRP, that maximizes asymptotical rate of growth:

$$\mathbf{b}^* := \underset{\mathbf{b}\in\Delta_d}{argmax}\,\mathbf{E}\{\ln\langle\mathbf{b},\mathbf{e}+\mathbf{R}\rangle\}.$$

The BCRP strategy is growth optimal in case of i.i.d. return distributions in a frictionless market (see Györfi et al., 2007).

Properties of Growth Optimal Investment

Besides maximization of asymptotic rate of growth, log-optimal investment has several other attractive properties:

• *Short term competitive optimality*: In case of known market distribution, the growth optimal strategy is optimal on the short term in the sense that

$$\mathbf{P(}S_t^* \geq S_t^{'}\mathbf{)} \geq 1/2\mathbf{,}$$

where S_t^* is wealth of the growth optimal portfolio and $S_t^{'}$ is the wealth achieved by any other investment strategy. This means that given two portfolio managers, the manager of a growth optimal portfolio probably outperforms the other (Bell & Cover, 1980).

• *Goal driven optimality*: Time required to achieve certain capital A is minimized by growth optimal investment in the limit, as $A \to \infty$ (Bell & Cover, 1980). It is suspected, that the probability of reaching a given level of wealth in a finite period is also maximized by the growth optimal portfolio (see Roll, 1977).

• *Short term proportional optimality*: From the Kuhn-Tucker characterization of growth optimal portfolio follows that

$$\mathbf{E}\left\{S_t^{'}/S_t^*\right\} \leq 1.$$

For more details see Luenberger (1998).

• *Consumption optimality*: According to Fama and MacBeth (1973) the growth optimal portfolio with consumption is also growth optimal.

• *Historical optimization property*: Simplicity of growth optimal investment can be seen on the following example. Consider empirical estimation of logarithmic growth on an i.i.d. market:

$$\mathbf{E}\{\ln\langle\mathbf{b},\mathbf{e}+\mathbf{R}\rangle\} \approx \frac{1}{T}\sum_{t=1}^{T}\ln\langle\mathbf{b},\mathbf{e}+\mathbf{r}_t\rangle = \frac{1}{T}\ln\prod_{t=1}^{T}\langle\mathbf{b},\mathbf{e}+\mathbf{r}_t\rangle = \frac{1}{T}\ln S_T.$$

Thus maximizing wealth historically is asymptotically growth optimal. While the example demonstrates an interesting aspect of the strategy, it is easy to overfit this way given insufficient number of historical observations.

Reverse Optimization of Growth Optimal Portfolio

We introduce growth optimal pricing model based on the assumption, that investors choose growth optimal portfolios in each investment period. We assume changing series of return distributions ($\mathbf{R}_t \in R_+^d$), and market participants optimizing based on homogeneous expectations about future market conditions. The optimal portfolio is determined according to:

$$\mathbf{b}_t := \underset{\mathbf{b}\in\Delta_d}{argmax}\,\mathbf{E}\{\ln\langle \mathbf{b}, \mathbf{e}+\mathbf{R}_t\rangle\}.$$

Roll (1973) and Cover (1984) introduced characterization of the growth optimal portfolio, which allows us reverse optimization of returns. At each investment period investors optimize their portfolios by minimizing

$$f_{\mathbf{R}_t}(\mathbf{b}) = -\mathbf{E}\ln\langle \mathbf{b}, \mathbf{e}+\mathbf{R}_t\rangle,$$

subject to

$$\langle \mathbf{b},\mathbf{e}\rangle = 1.$$

An equivalent formulation of this convex optimization problem follows from theory of Lagrange multipliers. We aim at minimizing

$$f_{\mathbf{R}_t}(\mathbf{b}) + \lambda(\langle \mathbf{b},\mathbf{e}\rangle - 1), \tag{3}$$

with regard to $\mathbf{b} \in R^d$ and $\lambda \in R$. Being at the global minimum means that the derivative of (3) is zero. For the optimal portfolio $\langle \mathbf{b}_t,\mathbf{e}\rangle = 1$, and the following equations hold for $i = 1,\ldots,d$:

$$-\mathbf{E}\frac{1+R_t^{(i)}}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle} + \lambda = 0. \tag{4}$$

This means

$$-\mathbf{E}\frac{b_t^{(i)}(1+R_t^{(i)})}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle} + b_t^{(i)}\lambda = 0,$$

summing of which implies

$$-\mathbf{E}\frac{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle} + \langle \mathbf{b}_t, \mathbf{e}\rangle \lambda = 0,$$

i.e. $\lambda = 1.$ Substituting λ in (4),

$$\mathbf{E}\frac{1+R_t^{(i)}}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle} = 1.$$

We prefer another form of the equation – in a similar manner to Roll (1973) –, using

$$Cov(R_t^{(i)}, \frac{1}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle}) = \mathbf{E}\frac{1+R_t^{(i)}}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle} - \mathbf{E}(1+R_t^{(i)})\mathbf{E}\frac{1}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle}.$$

From this it follows that

$$\mathbf{E}(1+R_t^{(i)}) = \frac{1 - Cov(R_t^{(i)}, \frac{1}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle})}{\mathbf{E}\frac{1}{\langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle}}.$$

Due to rationality and homogeneous expectations of market participants, they will choose to the same optimal portfolio. In a market composed of growth optimal investors, the global market portfolio is growth optimal, i.e. $1+R_t^{(M)} = \langle \mathbf{b}_t, \mathbf{e}+\mathbf{R}_t\rangle.$ Finally we can state the pricing equation of GOPM as follows:

$$\mathbf{E}(1+R_t^{(i)}) = \frac{1 - Cov(R_t^{(i)}, \frac{1}{1+R_t^{(M)}})}{\mathbf{E}\frac{1}{1+R_t^{(M)}}}. \qquad (5)$$

While the formula does not explicitely contain the risk free rate, in our empirical investigation we denominate value of assets and the market portfolio in bond ($r_t^{(f)}$).

EMPIRICAL RESULTS

The strength of our empirical tests lies in the fact that we focused on estimation of covariance. Inreasing accuracy was necessary because (5) is sensible to this quantity. In contrast to previous research (Roll, 1973; Fama & MacBeth, 1973), we use weekly data and exponentially weighted covariance estimates. As mentioned earlier, we also denominate returns for GOPM in risk free rate.

To verify expected return forecasts provided by our models, we use out-of-sample testing on non-overlapping weekly periods through our data. At time $t-1$ an investor's sense of returns $\mathbf{R}_t$ and $R_t^{(M)}$, is estimated by historical observations. Half-life of exponentially weighted volatility and covariances is 20 weeks. Estimations of $\mathbf{E}(R_t^{(M)})$, $\mathbf{E}\frac{1}{1+R_t^{(M)}}$, $\mathbf{E}(SMB_t)$, $\mathbf{E}(HML_t)$ and $\mathbf{E}(MOM_t)$ are based on averaging through $1,\ldots,t-1$.

Data

We use the CRSP value-weighted total return market index (CRSP-VW) of all New York Stock Exchange, American Stock Exchange and NASDAQ listed stocks as market proxy. We obtained S&P 500 constitutents from 1st of January 1967 to 26th of December 2008. The risk-free rate is the proxied by yields of one-month treasury-bills from Ibbotson and Associates. Small-minus-big, high-minus-low and momentum factor time series are obtained from Kenneth French's homepage.

Instead of dealing with noisy series of single assets, we create 32 portfolios out of 1411 stocks of S&P 500. First dividing our set of stocks to two cohorts by beta, we divide these again to two by $Cov(R^{(i)},\frac{1}{1+R^{(M)}})$, and so on by $s^{(i)}$, $h^{(i)}$ and $mom^{(i)}$. We create portfolios based on the resulted 32 cohorts. Backtesting of forecasts starts at 2nd of January 1970, in order to establish sensible historical estimates for expected returns.

Statistics for Comparison

The following two statistics are used to describe accuracy of any two models in comparison:

• *Portfolios statistic*: We calculate sum squared error of both models for each portfolio, along the 39-year long time series. We obtain 2x32 sum squared errors. A model is preferred over an other, if it outperforms in majority of the portfolios. While the portfolio statistic is a roboust tool for comparing models, it does not incorporate the magnitude of differences in squared errors.

• *SNSE statistic*: Relying solely on the previous statistic, it may happen that a model performs better in most of the cases, but at the rare exceptions it performs unreasonably bad. The SNSE statistic compares sum of normalized squared errors along all estimations. Since expected value minimizes squared error due to Steiner's theorem, this is the most relevant statistic regarding expected return estimations. We establish SNSE by normalizing time series of each asset by its empirical variance. This way estimation of assets with different volatilities have the same importance.

In case of the portfolios statistic, we establish p-values based on two-sided tail probability tests, assuming binomial distribution. To compare models based on SNSE, we construct two-sided paired tests on the null-hypothesis that

$$H_0:\{\sum_{t=1}^{T}\sum_{i=1}^{d}\frac{\mathbf{E}(E_{1,t}^{(i)}-R_t^{(i)})^2}{Var(R^{(i)})}-\frac{\mathbf{E}(E_{2,t}^{(i)}-R_t^{(i)})^2}{Var(R^{(i)})}=0\},$$

where $E_{1,t}^{(i)}$ and $E_{2,t}^{(i)}$ denotes forecasted returns of the two models, and $Var(R^{(i)})$ stands for empirical variance estimate of assets. Because estimation errors are serially correlated (see

Figure 1), we account for this correlation by splitting the the time series to 20 blocks. Based on these 20 blocks, we perform both two-sided paired block bootstrap and Student's t-test.

Results

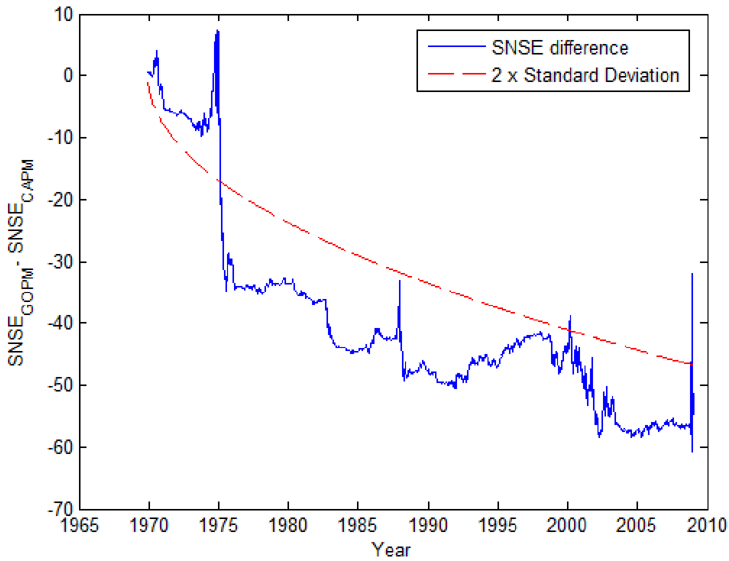
Our results establish statistically significant ordering between estimation methods, with the exception of Four-Factor Model versus CAPM. Models in order of increasing forecasting accuracy are historical average, Four-Factor Model, CAPM and GOPM.

Table 1: Statistical comparison of pricing models

Method	**Portfolios**	**P-value**	**SNSE diff.**	**Bootstrap p-value**	**Student's p-value**
Four-Factor - Average	30/32	0.00%	311	0.86%	2.15%
CAPM - Four-Factor	27/32	0.00%	104	31.67%	39.05%
GOPM - CAPM	31/32	0.00%	51	4.42%	3.33%

Comparisions between pairs of these methods can be found in Table 1. The portfolios column shows the number of cases where the first method outperforms the other out of the 32 test portfolios. P-values show solid statistical significances in each case. SNSE difference column shows the difference in sum of normalized squared errors. Differences are significant with the exception of CAPM versus Four-Factor Model. The fact that both CAPM and GOPM perform better than the Four-Factor Model may indicate change of factor exposures over time. For our most interesting test of GOPM versus CAPM, we also include evolution of SNSE difference on Figure 1.

Figure 1: Evolution of SNSE difference over time



Note that SNSE difference stays almost continuously under two standard deviations, in the statistically significant region. In contrast with Roll's (1973) and Fama and MacBeth's (1973) findings, our statistics reject the hypothesis, that CAPM based forecasts are as accurate as GOPM.

CONCLUSION

We compared three pricing models by forecast accuracy of expected returns. In contrast to earlier attempts of Roll (1973), and Fama and MacBeth (1973), we show significantly that the growth optimal pricing model provides superior expected value forecasts compared to CAPM and the Four-Factor Model. Success in doing so is a consequence of more accurate estimations of covariance using weekly data and exponential weighting.

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