Isoperimetric inequalities for Random Walks

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Abstract

In this paper some isoperimetric problems are studied, particularly the extremal property of the mean exit time of the random walk from finite sets. This isoperimetric problem is inserted into the set of equivalent conditions of the diagonal upper estimate of transition probability of random walks on weighted graphs.

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1 Introduction with some history

Isoperimetry is one of the oldest, if not the oldest, variational problem of mathematics. Zenodorus (lived not much later than Archimedes) is credited by Pappus to be the first mathematician studying the classical isoperimetric problem.

Among the fixed length simple closed planar curves the circle circumferences the biggest area.

This planar problem and its spacial analogue was already studied by Zenodorus with some success. It took almost two thousand years to rigorously prove this statement by Schwartz (1884) after the first modern steps taken by Steiner in 1836.

The importance of isoperimetric inequalities in physics was realized as early as 1877 by Lord Raleigh. He conjectured that the principal frequency of a membrane of fixed area is minimal for the circle. Several further quantities joined to this family including moment of inertia, electric capacity, torsional rigidity. For the history and modern foundation of the topic the reader is advised to [24],[23] or [7]. It turned out that isoperimetric inequalities are in close connection to ultra-contractive semigroup theory, particularly to diagonal upper estimates of transition probability of random walks ([30],

[31], [12], [7], [18],[1]) All these works demonstrate that functional analytic inequalities (like Sobolev, Nash or Poincaré inequality) and isoperimetric inequalities are also in strong connection.

We propose a new type of isoperimetric inequality.

Consider a measure metric space (Γ, μ, d) and assume that a diffusion is defined by the local, regular Dirichlet form on it (c.f. [15]). Let us fix a starting point $x \in \Gamma$ and define $E_x(T_A)$ as the mean exit time of the process starting at x from the set $A \subset \Gamma$. The following question can be raised.

Among the sets of fixed volume which one maximizes the mean exit time of the process?

We establish a partial answer to this question in the context of transient random walks on weighted graphs.

Defining an isoperimetric (or Faber-Krahn) function the problem inserted into the set of equivalent conditions of the diagonal upper estimate of transition probability of the random walk on a weighted graph.

Both results are new to our best knowledge. Similar isoperimetric result on the mean exit times where given in [20] [21] [22].

2 Basic definitions

Consider a countably infinite connected graph Γ with a weight function $\mu_{x,y} > 0$ on the edges $x \sim y$. This weight induces a measure $\mu(x)$

$$\mu(x) = \sum_{y \sim x} \mu_{x,y}$$

and

$$\mu(A) = \sum_{y \in A} \mu(y)$$

on the vertex sets $A \subset \Gamma$. It defines a reversible Markov chain as well, i.e. a random walk on the weighted graph (Γ, μ).We assume in the whole sequel that the transition probabilities are

$$P(x,y) = \frac{\mu_{x,y}}{\mu(x)} \geq p_0 \qquad (p_0)$$

uniformly for a fixed $p_0 > 0$. This implies that Γ has bounded geometry since $\deg(x) \leq 1/p_0$ and $\mu(A)$ is finite for finite sets. It is also evident that for all $x \sim y$ $\mu(x) \leq \frac{1}{p_0}\mu(y)$ and consequetly if $\overline{A} = \{y : \exists z \in A, y \sim z\}$ then $\mu\left(\overline{A}\right) \leq \frac{1}{p_0}\mu(A)$ as well.

Definition 2.1 *The random walk defined by (Γ, μ) will be denoted by (X_n).*

It is a reversible Markov chain on the state space Γ*, reversible with respect to the measure* μ*, has one step transition probability*

$$\mathbb{P}(X_{n+1} = y | X_n = x) = P(x, y) = \frac{\mu_{x,y}}{\mu(x)}$$

and the n*-step transition probability is*

$$P_n(x, y) = \mathbb{P}(X_n = y | X_0 = x).$$

For the sake of completeness we give a formal definition of the exit time.

Definition 2.2 *The exit time from a set* A *is defined as*

$$T_A = \min\{k : X_k \in \Gamma \backslash A\}$$

and its expected value is denoted by

$$E_x(A) = \mathbb{E}(T_A | X_0 = x).$$

The extreme mean exit time is defined as

$$\overline{E}(A) = \max_{x \in A} E_x(A)$$

Definition 2.3 *Denote* $H_t = H_{x,t} = \{y : G(x,y) \geq tG(x,x)\}$ *the super-level set of the Green's function*

$$G(x,y) = \sum_{k=0}^{\infty} P_k(x,y)$$

for $0 \leq t \leq 1$.

Definition 2.4 *For* $A \subset \Gamma$, $P^A = P^A(y,z) = P(y,z)|_{A\times A}$ *is a substochastic matrix (or the operator corresponding to the Dirichlet boundary condition), the restriction of* P *to the set* A. *Its iterates are denoted by* P_k^A *and it defines also a random walk killed at the exiting from the set.*

Definition 2.5 *We introduce*

$$G^A(y,z) = \sum_{k=0}^{\infty} P_k^A(y,z)$$

the local Green function, the Green function of the killed walk.

Definition 2.6 *For any two disjoint sets* $A, B \subset \Gamma$ *the resistance* $\rho(A,B)$ *is defined as*

$$\rho(A,B) = \inf\left\{((I-P)f, f)_\mu : f|_A = 1, f|_B = 0\right\}^{-1}$$

Let us mention that for a finite set $A, \rho_x(A) < \infty$ if and only if the RW is transient.

Remark 2.1 *It is well-known that (c.f. [25]) $x \in A \subset \Gamma$*

$$G^A(x,x) = \mu(x)\rho(\{x\}, \Gamma \backslash A).$$

Definition 2.7 *The Laplace operator of finite sets can be defined as $(I-P)|_{A\times A}$. The smallest eigenvalue of the Laplacian is denoted by $\lambda(A)$.*

Definition 2.8 *We shall say that the function $L(v)$ has doubling property if there is a $C_L > 1$ such that*

$$L(2v) \leq C_L L(v) \qquad (\mathrm{D}_1)$$

for all $v > 0$. Also we will say that L grows at least polynomially if there are $A > 1$, $\varepsilon > 0$ such that

$$L(Av) \geq (1+\varepsilon)\, L(v) \qquad (\mathrm{D}_2)$$

for all $v > 0$.

For unimportant small (< 1) constants we will use c and big (> 1) constants will be denoted by C. The by-product constants of the calculation

will be absorbed into one.

3 Isoperimetric inequalities

The following useful inequalities are well known (*c.f.* [29]).

Lemma 3.1 *For all weighted graphs and* $\emptyset \neq B \subset A \subset \Gamma$ *finite sets*

$$\lambda^{-1}(A) \leq C\overline{E}(A), \tag{λE}$$

$$\lambda(A)\mu(B)\rho(B, \Gamma \backslash A) \leq 1. \tag{$\lambda\rho\mu$}$$

In some of the proofs we shall use the refined wire model where edges are considered as homogeneous resistors of conductance $\mu_{x,y}$ (see [25]). Also edges are identified with the unit interval, resistances and weights are proportional to the length and in such a way a continuous measure metric space is in our possession. The harmonic functions, particularly the Green's functions extend linearly to the wires.

Definition 3.1 *Denote* $\overrightarrow{\Gamma} = \{\overrightarrow{(x,y)} : x, y \in \Gamma, x \sim y\}$ *the set of (arbitrarily) oriented edges of* Γ. $W = \overrightarrow{\Gamma} \times [0,1]$ *and for a function* h *and for an* $x \in \Gamma$

the equipotential surface of x is

$$\Gamma_x = \{w = \overrightarrow{(y,z)} \times \{\alpha\} \in W : (1-\alpha)\, h(y) + \alpha h(z) = h(x)\}$$

and similarly $\Gamma_w = \{s \in W : h(s) = h(w)\}$ *for a* $w \in W$.

This setup allows us to speak about equipotential surfaces and these surfaces are the boundaries of super-level sets. At some given points we shall refer to the objects of the refined model using the $(.)^r$notation like for a set H by H^r.

Definition 3.2 *The resistance between a finite set A and "infinity" is defined for any reference pont $x \in A$ by*

$$\rho_x(A) = \lim_{t \to \mathbf{0}} \rho(A, \Gamma \backslash H^r_{x,t}) \ . \tag{3.1}$$

where

$$H^r_{x,t} = \{w \in W : w = \overrightarrow{(y,z)} \times \alpha : G\,(y,x)\,(1-\alpha) + G\,(z,x)\,\alpha \geq tG\,(x,x)\}.$$

Proposition 3.2 *If for a graph satisfying (p_0) with a real function L with*

property (D_1),(D_2) *the inequality*

$$\rho(B,\Gamma\backslash A) \leq L\left(\mu(A)\right)\mu(B)^{-1} \qquad (\rho)$$

holds for all finite $\emptyset \neq B \subset A \subset \Gamma$ *then*

$$\overline{E}(A) \leq CL\left(\mu(A)\right). \qquad (E)$$

Proof. Let us fix an arbitrary $m > 1$ constant. Denote $k := \min(j : m^{j-1}\mu(x) < \mu(A) \leq m^j\mu(x)\}$ and A_j the super-level set of $G^A(x,.)$ of volume of $\mu\left(A_j\right)$:

$$m^{j-1}\mu\left(x\right) < \mu\left(A_j\right) \leq m^j\mu\left(x\right).$$

for $1 < j < k$, particularly let $A_0 = \{x\}$.

$$\begin{aligned} E_x(A) &= \sum_{y\in A} G^A\left(x,y\right) = \sum_{y\in A} G^A\left(y,x\right)\frac{\mu\left(y\right)}{\mu\left(x\right)} \qquad (3.2)\\ &= \sum_{i=1}^{k}\sum_{y\in A_i\backslash A_{i-1}} G^A\left(y,x\right)\frac{\mu\left(y\right)}{\mu\left(x\right)} + G^A\left(x,x\right). \end{aligned}$$

Now we recall that

$$G^A(y,x)\frac{1}{\mu(x)} = \rho(\Upsilon_y^r, \Gamma\backslash A),$$

where Υ_y^r is the equipotential surface of y (c.f. [25]). Since A_{i-1} is a super-level set Υ_y^r runs outside of A_{i-1} for $y \in A_i\backslash A_{i-1}$ and we get

$$G^A(y,x)\frac{1}{\mu(x)} = \rho(\Upsilon_y^r, \Gamma\backslash A) \leq \rho(A_{i-1}, \Gamma\backslash A). \tag{3.3}$$

Our estimation can be continued as follows:

$$E_x(A) \leq \sum_{i=1}^{k} \rho(A_{i-1}, \Gamma\backslash A)\left(\mu(A_i) - \mu(A_{i-1})\right) + \mu(x)\,\rho(\{x\}, \Gamma\backslash A) \tag{3.4}$$

and "integration by part" in the discrete setup can be done with the rearrangement of the sum to receive

$$\begin{aligned} E_x(A) &\leq \sum_{i=1}^{k} \rho(A_{i-1}, \Gamma\backslash A)\left(\mu(A_i) - \mu(A_{i-1})\right) + \mu(x)\,\rho(\{x\}, \Gamma\backslash A) \\ &\leq \sum_{i=1}^{k} \left(\rho(A_{i-1}, \Gamma\backslash A) - \rho(A_i, \Gamma\backslash A)\right)\mu(A_i) + \mu(x)\,\rho(\{x\}, \Gamma\backslash A) \\ &\leq \sum_{i=1}^{k} \rho(A_{i-1}, A_i)\mu(A_i) + \mu(x)\,\rho(\{x\}, \Gamma\backslash A). \end{aligned}$$

Finally we can use condition (ρ) to receive

$$\begin{aligned} E_x(A) &\leq C\sum_{i=1}^{k}\frac{L(\mu(A_i))}{\mu(A_{i-1})}\mu(A_i)+\mu(x)\,\rho(\{x\},\Gamma\backslash A)\leq C\sum_{i=0}^{k}L(\mu(A_i)) \\ &= C\sum_{i=0}^{k}L(m^i\mu(x))\overset{(D_2)}{\leq} CL(m^k\mu(x))\overset{(D_1)}{\leq} CL(\mu(A)) \end{aligned}$$

which proves the statement. ■

The main observation of the present paper is the following theorem.

Theorem 3.3 *Assume, that* (Γ,μ) *satisfies* (p_0) *and* L *has property* (D_1) *and* (D_2)*, then the following inequalities are equivalent*

$$\overline{E}(A)\leq CL(\mu(A)) \text{ for all finite } A\subset\Gamma \qquad (E)$$

$$\lambda(A)^{-1}\leq CL(\mu(A)) \text{ for all finite } A\subset\Gamma \qquad (\mathrm{FK})$$

$$\|f\|_2\, L\left(c'\frac{\|f\|_1}{\|f\|_2^2}\right)\leq C\,\|\nabla f\|_2 \text{ for all } f\in c_0(\Gamma) \qquad (\mathrm{N})$$

$$\rho(B,\Gamma\backslash A)\leq CL(\mu(A))\,\mu(B)^{-1} \text{ for all finite } \emptyset\neq B\subset A\subset\Gamma \qquad (\rho)$$

$$p_n(x,x) \leq C\frac{1}{f(n)}, \qquad \text{(DUE)}$$

where $f(t)$ is the solution of the equation

$$k = \int_0^{f(k)} \frac{L(S)}{s} ds.$$

Proof. The implications can be seen as indicated below.

$$(E) \overset{(E\lambda)}{\Longrightarrow} (FK) \overset{(\lambda\rho\mu)}{\Longrightarrow} (\rho) \overset{\text{Proposition 3.2}}{\Longrightarrow} (E).$$

The other loop containing (FK), (DUE), (N) is well known. (c.f. [11]). ■

4 Transient graphs

In this section we deal with transient random walks. With the aid of the Green's function we can formulate some results in the spirit of Carron ([7]). The following general isoperimetric inequality is the entry point of our investigation.

Let we introduce some notations. The sequence of super-level sets is

defined for an arbitrary but fixed $q \in (0, p_0^2)$ as

$$H_{x,i} = H_{x,q^i} = \left\{ y \in \Gamma : G(x,y) \geq q^i G(x,x) \right\}$$

and a is chosen to satisfy $\mu(H_{x,q^{a-1}}) < \mu(A) \leq \mu(H_{x,q^a})$.

Theorem 4.1 *For all transient weighted graphs satisfying* (p_0)

$$E_x(A) \leq C E_x(H_{a+1}) \tag{4.5}$$

where $C > 1$ *independent of* A.

This result can be interpreted as a weak isoperimetric inequality, showing extremal property of the super-level sets of the Green function. Based on this, we shall see that it is enough to verify one of the equivalent statements in Theorem 3.3 for super-level sets of the Green function.

Lemma 4.2 *For all random walks and for sets* $H \subset A \subset \Gamma$, *where* H *is super-level set of* $G^A(x,y)$

$$E_x(A) \geq \rho(H, \Gamma \backslash A) \mu(H).$$

Proof. The proof is straightforward from the definition of super-level sets and from the observation, that their boundary is an equipotential surface in

the refined model, which means that

$$\min_{y\in H}\frac{1}{\mu(x)}G^A(y,x) =: \frac{1}{\mu(x)}G^A(\underline{y},x) = \rho(\Upsilon^r_{\underline{y}},\Gamma\backslash A) \geq \rho(H,\Gamma\backslash A) \qquad (4.6)$$

$$\begin{aligned} E_x(A) &= \sum_{y\in A} G^A(x,y) \geq \sum_{y\in H}\frac{\mu(y)}{\mu(x)}G^A(y,x) \\ &\overset{(4.6)}{\geq} \sum_{y\in H}\mu(y)\rho(H,\Gamma\backslash A). \end{aligned}$$

■

As a particular consequence of Lemma 4.2 we obtain the following.

Lemma 4.3 *For transient weighted graphs satisfying* (p_0)

$$E_x(H_{x,a+1}) \geq cq^{a+1}G(x,x)\mu(H_a) \qquad (4.7)$$

where $q \in (0,p_0^2)$ *and* $c = \left(\frac{p_0}{q} - \frac{1}{p_0}\right)$.

Proof. For sake of simplicity we suppress x in the subscript of $H_i = H_{x,i}$. From Lemma 4.3 we have

$$E_x(H_{a+1}) \geq \rho(H_a,\Gamma\backslash H_{a+1})\mu(H_a),$$

hence we need the estimate of $\rho(H_a, \Gamma \backslash H_{a+1})$. The interpolation of the refined network implies that

$$\rho(H_a, \Gamma \backslash H_{a+1}) \geq \rho(H_a^r, \Gamma \backslash H_{a+1}^r)$$

and by shorting of the equipotential surface of H_{a+1}^r the resistances are unchanged hence we can calculate the resistance using

$$\rho(H_a^r, \Gamma \backslash H_{a+1}^r) = \rho(H_a^r, \Gamma \backslash H_N^r) - \rho(H_{a+1}^r, \Gamma \backslash H_N^r).$$

If $N \to \infty$, thanks to the monitonicity we have convergence and we obtain

$$\rho(H_a^r, \Gamma \backslash H_{a+1}^r) = \rho_x(H_a^r) - \rho_x(H_{a+1}^r). \tag{4.8}$$

We treat the new terms separately. For all $y \in H_a, s \sim y, s \notin H_a$

$$\mu(x)\, \rho_x(H_a^r) \geq \mu(x)\, \rho_x(\Upsilon_s^r) = G(s,x) \overset{(p_0)}{\geq} p_0 G(y,x) \geq p_0 q^q G(x,x), \tag{4.9}$$

on the other hand for $z \in H_{a+1}, v \notin H_{a+1}, z \sim v$

$$\mu(x)\, \rho_x(H_{a+1}^r) \leq \mu(x)\, \rho_x(\Upsilon_z^r) = G(z,x) \overset{(p_0)}{\geq} \frac{1}{p_0} G(v,x) < \frac{q^{a+1}}{p_0} G(x,x). \tag{4.10}$$

From estimates (4.9) and (4.10) we can receive the lower estimate

$$\rho(H_a^r, \Gamma \backslash H_{a+1}^r) \geq p_0 q^q G(x,x) - \frac{q^{a+1}}{p_0} G(x,x) \tag{4.11}$$

$$\geq \left(\frac{p_0}{q} - \frac{1}{p_0} \right) q^{a+1} G(x,x)$$

which results the statement if $q < p_0^2$. ■

Proof of Theorem 4.1. Let $a_i = \mu(A \cap (H_{x,i} \backslash H_{x,i-1}))$ and $h_i = \mu(H_{x,i}), i \geq 1$, $a_0 = \mu(x)$. The exit time from A can be decomposed using the sets $A_i = A \cap H_{x,i} \backslash H_{x,i-1}$ as in (3.2) to receive

$$E_x(A) \leq G(x,x) \sum_{i=0}^{\infty} q^{i-1} a_i. \tag{4.12}$$

Let us define the integer a for which $\mu(H_{a-1}) \leq \mu(A) < \mu(H_a)$ holds, furthermore denote

$$l_i = h_i - h_{i-1} \text{ if } 1 \leq i \leq a$$

$$l_0 = h_0 \text{ and}$$

$$l_i = 0 \text{ if } i \geq a+1.$$

One can see that $l_i \geq a_i$ if $0 \leq i \leq a$ and consequently

$$\sum_{i=0}^{\infty} a_i \leq \mu(H_a) = h_a = \sum_{i=0}^{a} l_i.$$

As in the proof of Lemma 4.3 using the potential surface of $y \in H_i \backslash H_{i-1}$ we have

$$\begin{aligned} \frac{1}{\mu(x)} G^{H_{a+1}}(y,x) &= \rho(\Upsilon_y^r, \Gamma \backslash H_{a+1}) \geq \rho(\Upsilon_y^r, \Gamma \backslash H_{a+1}^r) \qquad (4.13) \\ &= \rho(\Upsilon_y^r, \Gamma \backslash H_N^r) - \rho(H_{a+1}^r, \Gamma \backslash H_N^r) \end{aligned}$$

where $N > a+1$ and if $N \to \infty$ we obtain

$$\frac{1}{\mu(x)} G^{H_{a+1}}(y,x) \geq \rho(\Upsilon_y^r) - \rho\left(H_{a+1}^r\right)$$

Using exactly the same arguments have led to (4.11) we can derive that

$$\begin{aligned} \frac{1}{\mu(x)} G^{H_{a+1}}(y,x) &\geq \rho_x(\Upsilon_y^r) - \rho_x\left(H_{a+1}^r\right) \\ &\geq \frac{1}{\mu(x)} \left(q^i G(x,x) - \frac{1}{p_0} q^{a+1} G(x,x) \right). \end{aligned}$$

The mean exit time can be decomposed as follows.

$$\begin{aligned}\frac{1}{G(x,x)}E_x(H_{a+1}) &\geq \sum_{i=1}^{a+1}\min_{y\in H_i\setminus H_{i-1}}\frac{G^{H_{a+1}}(y,x)}{G(x,x)}\left(\mu(H_i)-\mu(H_{i-1})\right)\\ &\geq \sum_{i=1}^{a+1}\left(q^i-\frac{1}{p_0}q^{a+1}\right)l_i = q\sum_{i=1}^{a}l_iq^{i-1}-\frac{1}{p_0}q^{a+1}\mu(H_a),\end{aligned}$$

which means by Lemma 4.3 that

$$\frac{C}{G(x,x)}E_x(H_{a+1})\geq\sum_{i=0}^{a}l_iq^{i-1} \tag{4.14}$$

The next step is to combine (4.14) and (4.12) to receive

$$\frac{1}{G(x,x)}\left(CE_x(H_{a+1})-E_x(A)\right)\geq\sum_{i=0}^{\infty}\left(Cl_i-a_i\right)q^{i-1}$$

which can be estimated from below as follows.

$$\begin{aligned}&\sum_{i=0}^{\infty}\left(Cl_i-a_i\right)q^{i-1}=\sum_{i=0}^{\infty}\left(\left(C-1\right)l_i+l_i-a_i\right)q^{i-1}=\\ &=\sum_{i=0}^{\infty}\left(C-1\right)l_iq^{i-1}+\sum_{i=0}^{a}\left(l_i-a_i\right)q^{i-1}+\sum_{i=a+1}^{\infty}\left(l_i-a_i\right)q^{i-1}\\ &\geq\sum_{i=0}^{a}\left(C-1\right)l_iq^{i-1}-\sum_{i=a+1}^{\infty}a_iq^{i-1}\\ &\geq\left(C-1\right)q^{a-1}\mu(H_a)-q^a\mu(A)\\ &\geq q^a\mu(A)\left[q^{-1}\left(C-1\right)-1\right]\geq 0.\end{aligned}$$

If we choose $C \geq \frac{q+1}{q} + \varepsilon, \varepsilon > 0$, then

$$C' E_x(H_{a+1}) \geq E_x(A).$$

■

The same arguments lead to the following less concise but useful conclusion.

Proposition 4.4 *For all transient weighted graphs with property* (p_0)

$$E_x(A) \leq C E_x(M_{k+1}) + C' \rho(M_{k+1}) \mu(M_k) \tag{4.15}$$

where $C > 1$ *is independent of* A, *and* k *is chosen to satisfy* $\mu(M_{x,k-1}) < \mu(A) \leq \mu(M_{x,k})$ *and* $M_{x,i} - s$ *are the biggest possible super-level sets of* $G(x,y)$ *of volume* $\mu(H_{x,i}) \leq \mu(x) m^i$.

The next theorem is slight generalization of Theorem 3.3. It is based on assumptions only on the super-level sets of $G(x,y)$.

Theorem 4.5 *Assume that* (Γ, μ) *is transient, satisfies* (p_0) *and* L *has property* (D_1) *with* $C_L < 2$ *and* (D_2), *then the following inequalities are equivalent*

$$\overline{E}(H) \leq C L(\mu(H)) \text{ for all } H \tag{EH}$$

$$\lambda(H)^{-1} \leq CL\left(\mu(H)\right) \ \textit{for all } H \tag{FKH}$$

$$\rho(H) \leq CL\left(\mu(H)\right)\mu(H)^{-1} \ \textit{for all } H \tag{ρH}$$

$$p_n(x,x) \leq C\frac{1}{f(n)} \tag{DUE}$$

where $f(k)$ is the solution of

$$k = \int_0^{f(k)} \frac{L(S)}{s} ds.$$

Remark 4.1 *Let us remark here that the inequality (ρH) is generalization of Carron's [7] one:*

$$\mu(H) \leq \gamma^{-p}, \tag{4.16}$$

where $\gamma = \rho(H)$. If we substitute $L(v) = v^{1-\frac{1}{p}}$ into (ρH), the inequality (4.16) *follows. In this sense this theorem provides three equivalent isoperimetric inequality as condition for the diagonal upper estimate, i.e. conditions using not all finite sets but only super-level sets of the Green function.*

Remark 4.2 *The role of C_L needs some clarification. One can see easily*

that from (DUE) *it follows that*

$$\sum_{k=1}^{k=n} p_k(x,x) \leq C \sum_{k=1}^{k=n} k^{-\gamma}$$

if $\gamma = (\log_2 C_L)^{-1}$ *and consequently the random walk is transient if* $C_L < 2$. *The reverse implication can not be derived from* (DUE) *or from the equivalent conditions. Also the first three conditions based on the existence of super-level sets, hence the prior assumption of transience can not be dropped.*

Proof of Theorem 4.5. The proof is very similar to the previous ones. The inequality (FKH) follows from (EH) by $(\lambda\rho\mu)$ and (EH) follows from (ρH) just as in Proposition 3.2. To close the circle we have to show $(FKH) \Longrightarrow (\rho H)$. For any $x \in \Gamma$ consider $G(x,y)$ and the super-level sets, particularly let $M_i = M_{x,i}$ denote the set for which $2^{i-1}\mu(H) < \mu(M_i) \leq 2^i\mu(H)$. Here we can refer again to the refined wire model and to the property of potential levels, applying it to the surface of $M_i^r s$ it follows that

$$\rho(H) = \rho(H, \Gamma \backslash M_1^r) + \sum_{i=1}^{\infty} \rho(M_i^r, \Gamma \backslash M_{i+1}^r).$$

In order to return to nonrefined subsets we use the comparison of the values of the Green's function of neighboring vertices at the inner boundary as in

(4.9)and (4.10) at the outer boundary to obtain

$$\rho(M_i^r, \Gamma \backslash M_{i+1}^r) \leq C\rho(M_i, \Gamma \backslash M_{i+1}).$$

Using $(\lambda\rho\mu)$ it folows that

$$\begin{aligned} \rho(M_i, \Gamma \backslash M_{i+1}) &\leq \frac{1}{\lambda(M_{i+1})\,\mu(M_i)} \\ \overset{(FKH)}{\leq} C\frac{L(\mu(H_{i+1}))}{2^i\mu(H)} &\leq C\frac{C_L^{i+1}L(\mu(H))}{2^i\mu(H)}. \end{aligned}$$

Finally using the conditionon C_L we receive

$$\rho(H) \leq C\sum_{i=0}^{\infty}\frac{C_L^{i+1}L(\mu(H))}{2^i\mu(H)} \leq C\frac{L(\mu(H))}{\mu(H)}. \tag{4.17}$$

The link to the diagonal upper estimate is provided by Proposition 4.4 and we apply (4.17) to M_k.

$$\begin{aligned} E_x(A) &\leq CE_x(M_{k+1}) + C'\rho(M_{k+1})\mu(M_k) \\ &\leq CL(\mu(M_{k+1})) \overset{(D_1)}{\leq} CL(\mu(M_k)) \\ &\leq CL(\mu(A)) \end{aligned}$$

Now we can apply Theorem 3.3 to get (DUE) and all the equivalent statements there as well. Of course the return route from (DUE) via (E) to

(EH) is trivial, hence the proof is complete. ■

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