# Lower bounds for transition probabilities on graphs

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**Abstract**

In this paper two-sided estimate of the distribution of the exit time is shown together with off diagonal heat kernel lower bound for random walks on weighted graphs.

## 1 Introduction

Today a large amount of work is devoted to upper and two-sided estimate of heat kernels in different spaces (c.f. [6],[7],[8],[12],[16]) The main challenge in that is to find connection between structural properties of the space and the behavior of the heat kernel. The study of the heat kernel in $\mathbb{R}^n$ of course dates back to much earlier results among others to Moser [14],[15] and Aronson [1]. In these celebrated works and in the recent one's one or an other type of iteration takes place, particularly chaining arguments appear at many points. The present paper would like to demonstrate the incredible power of the chaining arguments and provide a new one which replaces Aronson's chaining argument for graphs to obtain heat kernel lower estimates. The new approach eliminates the condition on the volume growth. All what follows is in the discrete graph setting in discrete time, but one can see that most of the arguments carry over to the continuous case. Also it is generally believed that the majorty of the essential phenomena and difficulties related to diffusion are present in the discrete case.

In the course of the study of the pre-Sierpinski gasket ( c.f.[13] [2] and bibliography there) and other structures heat kernel estimates where given, which in the simples case has the form as follows:

$$p_n(x,y)+p_{n+1}(x,y)\geq \frac{1}{cV(x,n^{1/\beta})}\exp\left(-C\left(\frac{d^{\beta}(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right) \tag{1.1}$$

$$p_n(x,y)\leq \frac{1}{cV(x,n^{1/\beta})}\exp\left(-C\left(\frac{d^{\beta}(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right) \tag{1.2}$$

In [11] sufficient and necessary condition where given for it. The off diagonal lower estimates without their upper counterpart received less attention. This paper is intended to provide a contribution in this direction.

During the proof of the upper estimate an interesting side-result can be observed. The distribution of the exit time from a ball has an upper estimate under a particular condition. Consider $T_B$, the exit time from a ball $B=B(x,R)$. The expected value of $T_B$ is denoted by $E(x,R)=E(T_B|X_0=x)$ assuming that the starting point is $x$. On many fractals (or fractal type graph) the space-time scaling function is $R^{\beta}$, $E(x,R)\simeq R^{\beta}$, for a $\beta\geq 2$, and this property implies that

$$\mathbb{P}\left(T_B<n|X_0=x\right)\leq C\exp\left(-\left(\frac{R^{\beta}}{Cn}\right)^{\frac{1}{\beta-1}}\right). \tag{1.3}$$

This estimate was given first in [5] and later an independent proof was provided in [10] using also a chaining argument.

One might wonder about the condition which ensures the same (up to the constant) lower bound.

The other observation one may make is that almost all the known transition probability lower estimates similar to (1.1) are obtained using the (diagonal-) upper estimate (1.2). In the present paper we give lower estimates without using an upper one and without assumption on the volume growth. The main results are illustrated for the particular case $E(x,R)\simeq R^{\beta}$ postponing the general statements after the necessary definitions. If the elliptic Harnack inequality (see Definition (2.14)) holds , then for $n\geq R$

$$\mathbb{P}\left(T_{x,R}<n|X_0=x\right)\geq c\exp\left(-\left(\frac{R^{\beta}}{cn}\right)^{\frac{1}{\beta-1}}\right). \tag{1.4}$$

and

$$p_n(x,y)+p_{n+1}(x,y)\geq \frac{1}{CV(x,n^{1/\beta})r^{D}}\exp\left(-C\left(\frac{d^{\beta}(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right) \tag{1.5}$$

where $r = \left(\frac{n}{d(x,y)}\right)^{\frac{1}{\beta-1}}$ , $n \geq d(x,,y) \geq 0$ and $D$ is a fixed constant.

The results are new from several point of view. First of all lower estimates like (1.4) are new in this generality to our best knowledge and help to understand the tail behavior of the exit time. One should also observe that this lower estimates match with the upper one's (1.3) and (1.2) obtained from stronger assumptions. The key steps are given in Proposition 3.3 and 3.4 which help to control the probability to hit a nearby ball, which is usually more difficult than to control exit from a ball.

In Section 2 the necessary definitions are introduced. In Section 3 we give the general form and proof of (1.4). In Section 4 we show a heat kernel lower bound (better than 1.5) for very strongly recurrent walks and in Section 5 we show a result which contains (1.5) as a particular case.

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## 2 Basic definitions

In this section we give all the necessary definitions for our discussion. Let us consider an infinite connected graph $\Gamma$. We assume, for sake of simplicity, that there are no multiple edges and loops.

Let $\mu_{x,y} = \mu_{y,x} > 0$ be a symmetric weight function given on the edges $x \sim y$. These weights induce a measure $\mu(x)$

$$\begin{aligned} \mu(x) &= \sum_{y\sim x} \mu_{x,y}, \\ \mu(A) &= \sum_{y\in A} \mu(y) \end{aligned}$$

on the vertex sets $A \subset \Gamma$. The weights $\mu_{x,y}$ define a reversible Markov chain $X_n \in \Gamma$, i.e., a random walk on the weighted graph $(\Gamma, \mu)$ with transition probabilities

$$P(x,y) = \frac{\mu_{x,y}}{\mu(x)},$$
$$P_n(x,y) = \mathbb{P}(X_n = y | X_0 = x).$$

The transition "density" or heat kernel for the discrete random walk is defined as

$$p_n(x,y) = \frac{1}{\mu(y)} P_n(x,y).$$

To avoid parity problems we introduce

$$\widetilde{p}_n(x,y) = p_n(x,y) + p_{n+1}(x,y).$$

In many cases we will assume that the one step transition probabilities are uniformly separated from zero, i.e., there is a $p_0 > 0$ such that

$$P(x,y) \geq p_0 > 0 \tag{$p_0$}$$

for all $x \sim y, \quad x, y \in \Gamma$.

**Definition 2.1** *The graph is equipped with the usual (shortest path length) graph distance $d(x,y)$ and open metric balls are defined for $x \in \Gamma$, $R > 0$ as*

$$B(x,R) = \{y \in \Gamma : d(x,y) < R\}$$

*and its $\mu-$measure is denoted by $V(x,R)$*

$$V(x,R) = \mu(B(x,R)).$$

**Definition 2.2** *We use*

$$\overline{A} = \{y \in \Gamma : \exists x \in A, x \sim y\}$$

*for the closure of set $A$, Denote $\partial A = \overline{A} \backslash A$ and $A^c = \Gamma \backslash A$ the complement of $A$.*

Let us introduce the exit time $T_A$ for a set $A \subset \Gamma$.

**Definition 2.3** *The exit time from a set $A$ is defined as*

$$T_A = \inf\{t \geq 0 : X_n \in A^c\},$$

*its expected value is denoted by*

$$\begin{aligned} E_y(A) &= \mathbb{E}(T_A | X_0 = y), \\ E_y(x,R) &= E_y(B(x,R)) \end{aligned}$$

*and we will use the $E = E(x,R) = E_x(B(x,R))$ short notations.*

The definition implies that $E(x,1) = 1$.

**Definition 2.4** *The hitting time $\tau_A$ of a set $A \subset \Gamma$ is defined by*

$$\tau_A = T_{A^c}.$$

**Definition 2.5** *We introduce the maximal exit time for $x \in \Gamma, R > 0$ by*

$$\overline{E}(x,R) = \max_{y \in B(x,R)} E_y(x,R).$$

**Definition 2.6** *One of the key assumptions in our study is the condition $(\overline{E})$: there is a $C > 0$ such that for all $x \in \Gamma, R > 0$*

$$\overline{E}(x,R) \le CE(x,R) \tag{2.6}$$

*is true.*

**Definition 2.7** *We say that the* time comparison principle *holds for $(\Gamma, \mu)$ if there is a $C_T > 1$ constant such that for any $x \in \Gamma, R > 0, y \in B(x,R)$*

$$\frac{E(y,2R)}{E(x,R)} \le C_T. \tag{2.7}$$

**Proposition 2.1** *From the time comparison principle it follows that*

$$\frac{E(x,2R)}{E(x,R)} \le C_T, \tag{2.8}$$

$$\overline{E}(x,R) \le CE(x,R) \tag{2.9}$$

*and there is a constant $A_T$ such that for all $x \in \Gamma, R > 0$*

$$E(x, A_T R) \ge 2E(x,R). \tag{2.10}$$

For the easy proofs see [17]. One can deduce from (2.8) that there is a $\beta > 1$ and $C > 0$ such that for all $R > S > 0$

$$\frac{E(x,R)}{E(x,S)} \le C\left(\frac{R}{S}\right)^{\beta},$$

$$E(x,R) \le CR^{\beta}$$

and from (2.10) that there is a $\delta > 0, c > 0$ such that

$$E(x,R) \ge cR^{\delta}.$$

**Definition 2.8** *For a given $x \in \Gamma, n \ge R > 0$ let us define $k = k(x,n,R)$ as the maximal integer for which*

$$\frac{n}{k} \le q \min_{z \in B(x,R)} E\left(z, \frac{R}{k}\right)$$

*where $q$ is a fixed constant. Let $k = 1$ by definition if there is no such integer.*

**Definition 2.9** *Let us denote the set of vertices in the shortest path connecting* $x$ *and* $y$ *by* $\pi_{x,y}$.

**Definition 2.10** *For* $x, y \in \Gamma, n \geq R > 0, C > 0$ *let us define* $l = l_C(x, y, n, R)$ *as the minimal integer for which*

$$\frac{n}{l} \geq Q \max_{z \in \pi_{x,y}} E\left(z, \frac{CR}{l}\right)$$

*where* $Q$ *is a fixed constant (to be specified later.), Let* $l = R$ *by definition if there is no such an integer. If* $d(x, y) = R$ *we will use the shorter notation* $l_C(x, y, n) = l_C(x, y, n, d(x, y))$

**Definition 2.11** *For a given* $x \in \Gamma, n \geq R > 0$ *let us define*

$$\nu = \nu(x, n, R) = \min_{y \in B(x,R)^c} l_9(x, y, n, R).$$

**Definition 2.12** *In general,* $a_\xi \simeq b_\xi$ *will mean that there is a* $C > 0$ *such that for all* $\xi$

$$\frac{1}{C} a_\xi \leq b_\xi \leq C a_\xi.$$

**Definition 2.13** *A function* $h : \Gamma \to \mathbb{R}$ *said to be harmonic on* $A \subset \Gamma$ *if it is defined on* $\overline{A}$ *and*

$$\sum_{y \in \Gamma} P(x, y) h(y) = h(x) \quad \text{for all } x \in A. \tag{2.11}$$

**Definition 2.14** *We say that the weighted graph* $(\Gamma, \mu)$ *satisfies* $(H)$ the elliptic Harnack inequality *if there is a constant* $C > 0$ *such that for all* $x \in \Gamma, R > 0$ *and for any non-negative harmonic function* $u$ *which is harmonic in* $B(x, 2R)$*, the following inequality holds*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u. \tag{2.12}$$

If the weights of the edges are considered as wires, the whole graph can be seen as an electric network. Resistances are defined using the usual capacity notion.

**Definition 2.15** *On $(\Gamma, \mu)$ the Dirichlet form is defined as*

$$\mathcal{E}(f, f) = \sum_{y \sim z} \mu_{y,z} (f(y) - f(z))^2$$

*and the inner product is*

$$(f, f) = \sum_{y} f^2(x) \mu(x).$$

**Definition 2.16** *For any sets $A, B$ the capacity is defined via the Dirichlet form $\mathcal{E}$ by*

$$cap(A, B) = \inf \mathcal{E}(f, f),$$

*where the infimum runs for functions $f$, $f|_A = 1, f|_B = 0$. The resistance is defined then as*

$$\rho(A, B) = \frac{1}{cap(A, B)}.$$

*In particular we will use the following notations:*

$$\rho(x, r, R) = \rho(B(x, r), B^c(x, R)).$$

# 3 Distribution of the exit time

In this section we prove the following theorem.

**Theorem 3.1** *Assume that the weighted graph $(\Gamma, \mu)$ satisfies $(p_0)$.*
*1. If $(\overline{E})$ holds , then there are $c, C > 0$ such that for all $n \geq R > 0, x \in \Gamma$*

$$P(T_{x,R} < n) \leq C \exp(-ck(x, n, R))$$

*is true.*

*2. If $(\Gamma, \mu)$ satisfies the elliptic Harnack inequality $(H)$, then there are $c, C > 0$ such that $n \geq R > 0, x \in \Gamma$*

$$P(T_{x,R} < n) \geq c \exp(-C\nu(x, n, R)). \tag{3.13}$$

The proof of the upper bound was given in [17]. The lower bound is based on a chaining argument. First we need some propositions.

**Proposition 3.2** *Assume that the weighted graph* $(\Gamma, \mu)$ *satisfies* $(p_0)$ *and* $(\overline{E})$ *, then there is a* $c > 0$ *such that for all* $x \in \Gamma, n, R > 0$

$$P(T_{x,R} > n) > c,$$

*if* $n \leq \frac{1}{2} E(x, R)$

**Proof.** For the proof see Lemma 5.3 of [17] ■

**Proposition 3.3** *Assume that the weighted graph* $(\Gamma, \mu)$ *satisfies* $(p_0)$ *and* $(H)$*. Then there are* $c_0, c_1 > 0$ *such that for all* $x, y \in \Gamma, r > 0, d(x, y) < 4r, m > \frac{2}{c_1} E(x, 9r)$

$$P_x(\tau_{y,r} < m) > c_0$$

**Proposition 3.4** *Assume that the weighted graph* $(\Gamma, \mu)$ *satisfies* $(p_0)$ *and the elliptic Harnack inequality* $(H)$*. Then there are* $c, C, C' > 0$ *such that for all* $x, y \in \Gamma, r \geq 1,\ n > d(x, y) - r$

$$P_x(\tau_{y,r} < n) \geq c \exp\left[-C' l_C(x, y, n, d(x, y) - r)\right].$$

**Lemma 3.5** *If* $(\Gamma, \mu)$ *satisfies* $(p_0)$ *and the elliptic Harnack inequality* $(H)$*, then for* $x \in \Gamma, r > 0, K > L \geq 1, B = B(x, Kr), S = \{y : d(x, y) = Lr\}$

$$\min_{w \in S} g^B(w, x) \simeq \rho(x, Lr, Kr) \simeq \max_{v \in S} g^B(v, x). \tag{3.14}$$

**Proof.** See Barlow's proof ([4], Proposition 2) which generalizes Propositions 4.1 and 4.3 of [11] where the additional hypothesis of bounded covering was used. ■

**Lemma 3.6** *If* $(\Gamma, \mu)$ *satisfies* $(p_0)$ *and the elliptic Harnack inequality* $(H)$*, then there is a* $c_1 > 0$ *such that for all* $x \in \Gamma, r > 0, w \in B(x, 4r)$

$$\mathbb{P}_w(\tau_{x,r} < T_{x,5r}) > c_1. \tag{3.15}$$

**Proof.** The investigated probability

$$u(w) = \mathbb{P}_w(\tau_{x,r} < T_{x,5r}) \tag{3.16}$$

is the capacity potential between $\Gamma \backslash B(x, 5R)$ and $B(x, R)$ and clearly harmonic in $A = B(x, 5R) \backslash B(x, R)$. So it can be as usual decomposed

$$u(w) = \sum_z g^{B(x,5R)}(w, z) \pi(z)$$

with the proper capacity measure $\pi(z)$ with $\pi(A) = 1/\rho(x, R, 5R)$. From the maximum (minimum) principle it follows that $w \in S(x, 4R-1)$ and from the Harnack inequality for $g^{B(x,5R)}(w, .)$ in $B(x, 2R)$ that

$$\min_{z \in \overline{B}(x,R+1)} g^{B(x,5R)}(w, z) \geq c g^{B(x,5R)}(w, x)$$

$$u(w) = \sum_z g^{B(x,5R)}(w, z)\pi(z) \geq \frac{c g^{B(x,5R)}(w, x)}{\rho(x, R, 5R)}$$

From Lemma 3.5 we know that

$$\max_{y \in B(x,5R) \setminus B(x,4R)} g^{B(x,5R)}(y, x) \simeq \rho(x, 4R, 5R) \simeq \min_{w \in B(x,4R)} g^{B(x,5R)}(w, x).$$

which means that

$$u(w) \geq c \frac{\rho(x, 4R, 5R)}{\rho(x, R, 5R)} \tag{3.17}$$

Similarly from Lemma 3.5 it follows that

$$\max_{y \in B(x,5R) \setminus B(x,R)} g^{B(x,5R)}(v, x) \simeq \rho(x, R, 5R) \simeq \min_{w \in B(x,R)} g^{B(x,5R)}(w, x).$$

Finally if $y_0$ is on the ray from $x$ to $y$ then iterating the Harnack inequality along a finite chain of balls of radius $R/4$ along this ray from $y_0$ to $y$ one obtains

$$g^{B(x,5R)}(y, x) \simeq g^{B(x,5R)}(y_0, x)$$

which results that

$$\rho(x, R, 5R) \geq c\rho(x, 4R, 5R),$$

and the statement follows from $(3.17)$. ■

Now we are ready to give the proof of Proposition 3.3.

**Proof of Proposition 3.3.** We insert the exit time $T_{x,9r}$ into the inequality $\tau_{z,r} < m$

$$\begin{aligned}
\mathbb{P}_x(\tau_{z,r} < m) &\geq \mathbb{P}_x(\tau_{z,r} < T_{x,9r} < m) \\
&= \mathbb{P}_x(\tau_{z,r} < T_{x,9r}) - \mathbb{P}_x(\tau_{z,r} < T_{x,9r}, T_{x,9r} \geq m) \\
&\geq \mathbb{P}_x(\tau_{z,r} < T_{x,9r}) - \mathbb{P}_x(T_{x,9r} \geq m).
\end{aligned}$$

On one hand

$$\mathbb{P}_x(T_{x,9r} \geq m) \leq \frac{E(x, 9r)}{m} \leq \frac{E(x, 9r)}{\frac{2}{c_1}E(x, 9r)} < c_1/2$$

and on the other hand $B(z, 5r) \subset B(x, 9r)$, hence

$$\mathbb{P}_x(\tau_{z,r} < T_{x,9r}) \geq \mathbb{P}_x(\tau_{z,r} < T_{z,5r}),$$

and Lemma 3.6 can be applied to get

$$\mathbb{P}_x(\tau_{z,r} < T_{z,5r}) \geq c_1.$$

The result follows with $c_0 = c_1/2$. ■

**Lemma 3.7** *Let us assume that $x \in \Gamma, m, r, l \geq 1$, denote $n = ml, 0 \leq u \leq 3l, R = (3l-2)r - u, y \in S(x, R+r)$, then*

$$P_x(\tau_{y,r} < n) \geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z^l(\tau_{w,r} < m).$$

*where $\pi_{x,y}$ is the union of vertices of shortest paths from $x$ to $y$.*

**Proof.** We define a chain of balls. For $1 \leq l \leq d(x,y) - r$ let us consider a sequence of vertices $x_0 = x, x_1, \ldots x_l = y, x_i \in \pi_{x,y}$ in the following way: $d(x_{i-1}, x_i) = r - \delta_i$, where $\delta_i \in \{0,1,2,3\}$ for $i = 1 \ldots l$ and

$$u = \sum_{i=1}^{l} \delta_i$$

$$R = (3l-2)r - \sum_{i=1}^{l} \delta_i = (3l-2)r - u.$$

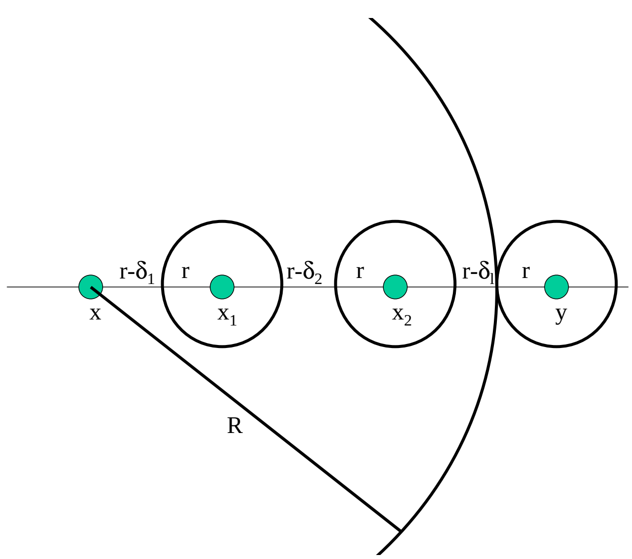


**Figure 1** Chain of balls

$\tau_i = \tau_{x_i,r}$ and $A_i = \{\tau_i - \tau_{i-1} < m\}$ for $i = 1, ...l, \tau_0 = 0$. One can observe that $\prod_{i=1}^{l} A_i$ means that the walk takes less than $m$ steps between the first hit of the consecutive $B_i = B(x_i, r)$ balls, consequently

$$P_x(\tau_{y,r} < n) \geq \mathbb{E}_x\left(\prod_{i=1}^{l} I(A_i)\right)$$

From this one obtains the following estimates denoting $z_0 = x$

$$\begin{aligned}
\mathbb{P}_x(\tau_{y,r} < n) &\geq \mathbb{E}_x\left(\prod_{i=1}^{l} I(A_i)\right) \qquad (3.18)\\
&= \mathbb{E}_x\left(\prod_{i=1}^{l} \sum_{z_i \in \partial B_i} I(\tau_i < m, X_{\tau_i} = z_i)\right)\\
&= \mathbb{E}_x\left(\sum_{z_1 \in \partial B_1} \sum_{z_2 \in \partial B_2} .. \sum_{z_l \in \partial B_l} \prod_{i=1}^{l} I(\tau_i < m, X_{\tau_i} = z_i)\right)\\
&= \sum_{z_1 \in \partial B_1} \sum_{z_2 \in \partial B_2} .. \sum_{z_l \in \partial B_l} \mathbb{E}_x\left[\prod_{i=1}^{l} I(\tau_i < m, X_{\tau_i} = z_i)\right]\\
&\overset{*}{=} \sum_{z_1 \in \partial B_1} \sum_{z_2 \in \partial B_2} .. \sum_{z_l \in \partial B_l} \prod_{i=1}^{l} \mathbb{P}_{z_{i-1}}(\tau_i < m) \mathbb{P}(X_{\tau_i} = z_i)\\
&\geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) = 4r} \mathbb{P}_z^l(\tau_{w,r} < m),
\end{aligned}$$

where in the $\overset{*}{=}$ step the Markov property was used. ■

Now we can prove the main ingredient of this section, which helps to control the probability of a nearby ball.

**Proof of Proposition 3.4.** If $n > \frac{2}{c_1} E(x, 9R)$, then the statement follows from Proposition 3.3. Also if $r \leq 9$ , then $\frac{R}{3r} \leq l \leq R$, so from $(p_0)$ the trivial lower estimate

$$P_x(\tau_{y,r} < n) \geq c \exp\left(-27\left(\log\frac{1}{p_0}\right) l\right)$$

gives the statement. If $n < \frac{2}{c_1} E(x, 9R)$ and $r \geq 10$, then $l_9(x, y, n, R) > 1$ and $R = (3l-2)r - u \geq 34$. Let us use Proposition 3.3 and Lemma 3.7. The latter one states that

$$P_x(\tau_{y,r} < n) \geq \min_{w \in \pi_{x,y}, 2r-3 \leq d(z,w) \leq 4r} \mathbb{P}_z^l(\tau_{w,r} < m).$$

and by Proposition 3.3 $\mathbb{P}^l_z(\tau_{w,r}<m)>c$ if for $w\in\pi_{x,y}$

$$\frac{n}{l}>\frac{2}{c_1}E(w,9r)=\frac{2}{c_1}E\left(w,9\frac{R+u}{3l-2}\right). \tag{3.19}$$

Consider the following straightforward estimates for $r\geq 10, R\geq 10$.

$$\begin{aligned} 9r &= 10(r-1)\leq 10\left(\frac{R+u}{3l-2}-1\right)\leq 10\left(\frac{R+3l}{3l-2}-1\right)=10\frac{R+2}{3l-2} \\ &\leq \frac{4R}{(l-1)}\leq 8\frac{R}{l}<9\frac{R}{l}. \end{aligned}$$

If $l=l_9(x,y,n,R)$, then the (3.19) inequality is satisfied and Proposition 3.3 can be applied to get uniform lower bound for all $\mathbb{P}^l_z(\tau_{w,r}<m)$. ■

**Proof of Theorem 3.1.** The upper estimate of Theorem 3.1 can be seen along the lines of the proof of Theorem 5.1 in [17]. The lower bound is immediate from Proposition 3.4 minimizing $l_9(x,y,n)$ for $y\in B(x,R)^c$ and $r=d(x,y)-R>1$, because for any $y\in B(x,R)^c$

$$\mathbb{P}_x(T_{x,R}<n)\geq\mathbb{P}_x(\tau_{y,r}<n).$$

■

# 4 Very strongly recurrent graphs

**Definition 4.1** *Following [2] we say that a graph is very strongly recurrent ($VSR$) if there is a $c>0$ such that for all $x\in\Gamma, r>0, w\in\partial B(x,r)$*

$$\mathbb{P}_w(\tau_x<T_{x,2r})\geq c.$$

In this section we deduce an off-diagonal heat kernel lower bound for very strongly recurrent graphs. The proof is based on Theorem 3.1 and the fact that very strong recurrence implies the elliptic Harnack inequality (c.f. [2]). Let us mention here that the strong recurrence was defined among others in [17] and one can easily see that strong recurrence in conjunction with the elliptic Harnack inequality is equivalent with very strong recurrence. It is worth to note, that the usually considered finitely ramified fractals and their prefractal graphs are (very) strongly recurrent.

**Theorem 4.1** *Let us assume that $(\Gamma,w)$ satisfies $(p_0)$ and is very strongly recurrent furthermore satisfies $\left(\overline{E}\right)$. Then there are $b,c>0$ such that for all $x,y\in\Gamma, n>0$*

$$\widetilde{p}_n(x,y)\geq\frac{c}{V(x,e(x,n))}\exp\left[-Cl_9\left(x,y,\frac{1}{2}n,d\right)\right], \tag{4.20}$$

*where $d=d(x,y)$.*

**Remark 4.1** *Typical examples for very strongly recurrent graphs are pre-fractal skeletons of p.c.f. self similar sets (for the definition, and further reading see [2] and [3]). We recall Barlow's [2] and Delmotte's [9] constructions. Let us consider* $\Gamma_1, \Gamma_2$ *two trees which are* $(VSR)$ *and assume that* $V_i(x,R) \simeq R^{\alpha_i}, E(x,R) \simeq R^{\beta_i}$, $\alpha_1 \neq \alpha_2$,

$$\gamma = \beta_1 - \alpha_1 = \beta_2 - \alpha_2 > 0$$

*which basically means that*

$$\rho(x,R,2R) \simeq R^{\gamma}$$

*for both graphs. Such trees are constructed in [2]. Let* $\Gamma$ *be the joint of* $\Gamma_1$ *and* $\Gamma_2$*, which means that two vertices* $O_1, O_2$ *are chosen and identified (for details see [9]). The resulting graph satisfies the Harnack inequality but not the volume doubling property. One can also see that* $\Gamma$ *is a* $(VSR)$ *tree as well using the fact that* $\Gamma_i - s$ *are trees, . This means that* $\Gamma$ *is an example for graphs that satisfies the Harnack inequality but not the usual volume properties.*

It was realized some time ago that the so-called near diagonal lower bound (4.21) is a crucial step to obtain off-diagonal lower estimates. Here we utilize the fact that the near diagonal lower bound is an easy consequence of very strong recurrence. As we shall see the proof does not use the diagonal upper estimate and assumption on the volume.

**Proposition 4.2** *Assume* $(p_0)$ *and* $\left(\overline{E}\right)$*, then there is a* $c > 0$ *such that for all* $x \in \Gamma, n \geq 0$

$$p_{2n}(x,x) \geq \frac{c}{V(x, e(x,2n))},$$

*where* $e(x,.)$ *is the inverse of* $E(x,.)$ *in the second variable.*

For the proof see [17].

**Proposition 4.3** *Let us assume that* $(\Gamma, \mu)$ *satisfies* $(p_0)$*. If the graph is very strongly recurrent,* $\left(\overline{E}\right)$ *holds, then there are* $c, c' > 0$ *such that for all* $x, y \in \Gamma, m \geq \frac{2}{c'} E(x, 2d(x,y))$

$$\widetilde{p}_m(x,y) \geq \frac{c}{V(x, e(x,m))}. \tag{4.21}$$

**Proof.** The proof starts with a first hit decomposition and uses Proposition 4.2.

$$\begin{aligned}\widetilde{p}_m(y,x) &\geq \sum_{i=0}^{m-1}\mathbb{P}_y(\tau_x=i)\,\widetilde{p}_{m-i}(x,x)\geq \mathbb{P}_y(\tau_x<m)\,\widetilde{p}_m(x,x)\\ &\geq \frac{c}{V(x,e(x,m))}\mathbb{P}_y(\tau_x<m).\end{aligned}$$

We estimate the latter term as in the proof of Proposition 3.3. Denote $r=d(x,y)$,

$$\mathbb{P}_y(\tau_x<m)\geq \mathbb{P}_y(\tau_x<T_{x,2r}<m)\geq \mathbb{P}_y(\tau_x<T_{x,2r})-\mathbb{P}_y(T_{x,2r}\geq m).$$

From $(VSR)$ we have that $\mathbb{P}_y(\tau_x<T_{x,2r})>c$ so from $m\geq\frac{2}{c'}E(x,2r)$ and from the Markov inequality it follows that

$$\mathbb{P}_y(T_{x,2r}\geq m)\leq\frac{E(x,2r)}{m}\leq c'/2.$$

Consequently we have that $\mathbb{P}_y(\tau_x<s)>c'/2$ and the result follows. ■

**Proof of Theorem 4.1.** If $l=l_9(x,y,n,d(x,y))=1$ , then $n>\frac{2}{c'}E(x,9d)>\frac{2}{c'}E(x,2d)$ and the statement follows from Proposition 4.3. Let us assume that $l>1$ and start with a path decomposition. Denote $m=\left\lfloor\frac{n}{l}\right\rfloor$, $r=\left\lfloor\frac{R}{l}\right\rfloor$, $S=\{y:d(x,y)=r\}$, $\tau=\tau_S$

$$\begin{aligned}\widetilde{p}_n(y,x) &= \frac{1}{\mu(x)}\mathbb{P}_y(X_n=x\text{ or }X_{n+1}=x\ )\\ &\geq \sum_{i=0}^{n-m-1}\sum_{w\in S}\mathbb{P}_y(X_\tau=w,\tau=i)\min_{w\in S}\widetilde{p}_{n-i}(w,x)\\ &\geq \sum_{i=0}^{n-m-1}\mathbb{P}_y(\tau=i)\min_{w\in S}\widetilde{p}_{n-i}(w,x).\end{aligned}$$

The next step is to use the near diagonal lower estimate:

$$\begin{aligned}\widetilde{p}_n(y,x) &\geq \sum_{i=0}^{n-m-1}\mathbb{P}_y(\tau=i)\min_{w\in S}\widetilde{p}_{n-i}(w,x)\\ &\geq \sum_{i=0}^{n-m-1}\mathbb{P}_y(\tau=i)\frac{c}{V(x,e(x,n-i))}\\ &\geq \mathbb{P}_y\left(\tau<\frac{n}{2}\right)\frac{c}{V(x,e(x,n))}.\end{aligned}$$

In the proof of Theorem 3.1 we have seen that

$$\mathbb{P}_y\left(\tau_{x,r}<\frac{n}{2}\right)\geq c\exp -Cl_9\left(x,y,\left(\frac{n}{2}\right),d-r\right),$$

which finally gives that

$$\begin{aligned}\widetilde{p}_n(y,x) &\geq \frac{c}{V(x,e(x,n))}\exp -Cl_9\left(x,y,\frac{n}{2},d-r\right)\\ &\geq \frac{c}{V(x,e(x,n))}\exp -Cl_9\left(x,y,\frac{1}{2}n,d\right).\end{aligned}$$

■

# 5 Heat kernel lower bound for graphs

In this section the following off-diagonal lower bound is proved.

**Theorem 5.1** *Let us assume that the graphs* $(\Gamma,\mu)$ *satisfies* $(p_0)$. *Also we suppose that the time comparison principle* (2.7) *and the elliptic Harnack inequality* $(H)$ *hold. Then there are* $c,C,D>0$ *constants such that for any* $x,y\in\Gamma, n\geq d(x,y)$

$$\widetilde{p}_n(x,y)\geq\frac{c}{V(x,e(x,n))\,r^D}\exp\left(-Cl_9\left(x,y,\frac{n}{2}\right)\right) \tag{5.22}$$

*where* $e(x,n)$ *is the inverse of* $E(x,R)$ *in the second variable and* $l = l_9\left(x,y,\frac{n}{2}\right)$, $R=d(x,y)$, $r=\frac{R}{3l}$.
*In particular if* $n<c\frac{E(x,R)}{(\log E(x,R))^{\beta-1}}$, *then*

$$\widetilde{p}_n(x,y)\geq\frac{c}{V(x,e(x,n))}\exp\left(-Cl_9\left(x,y,\frac{n}{2}\right)\right). \tag{5.23}$$

**Corollary 5.2** *If we assume* $(p_0)$ *and* $E(x,R)\simeq R^\beta$ *and that the elliptic Harnack inequality* $(H)$ *is true, then*

$$\widetilde{p}_n(x,y)\geq\frac{c}{V\left(x,n^{\frac{1}{\beta}}\right)}\exp\left(-C\left[\frac{R^\beta}{n}\right]^{\frac{1}{\beta-1}}\right).$$

*for* $n<c\frac{R^\beta}{(\log E(x,R))^{\beta-1}}$ *where* $R=d(x,y)$.

This corollary is a direct consequence of Theorem 5.1.

**Proposition 5.3** *Let us assume that* (2.10) *and the Harnack inequality* $(H)$ *holds. Then there are* $D, c > 0$ *such that for* $x, y \in \Gamma$, $r = d(x, y)$, $m > CE(x, r)$ *the inequality*

$$\widetilde{p}_m(y, x) \geq \frac{c}{V(x, e(x, m))} r^{-D}$$

*holds.*

**Proof.** The proof is based on a modified version of the chaining argument used in the proof of Lemma 3.7. From Proposition 4.3 we know that $(\overline{E})$ implies

$$\widetilde{p}_n(x, x) \geq \frac{c}{V(x, e(x, n))}. \tag{5.24}$$

Let us recall (2.10) and set $A = \max\{9, A_T\}$. Consider a sequence of times $m_i = \frac{m}{2^i}$ and radii $r_i = \frac{r}{A^i}$. From the condition $m > CE(x, r)$ and (2.10) it follows that for all $i$

$$m_i > CE(x, r_i) \tag{5.25}$$

holds as well. Let us denote $B_i = B(x, r_i), \tau_i = \tau_{B_i}$ and start a chaining.

$$\begin{aligned} \widetilde{p}_m(y, x) &= \sum_{k=1}^{m} P(\tau_1 = k) \min_{w \in \partial B_1} \widetilde{p}_{m-k}(w, x) \geq \sum_{i=1}^{m/2} P(\tau_1 = k) \min_{w \in \partial B_1} \widetilde{p}_{m-k}(w, x) \\ &\geq P(\tau_1 < m/2) \min_{1 \leq k \leq m/2} \min_{w \in \partial B_1} \widetilde{p}_{m-k}(w, x). \end{aligned}$$

Let us continue in the same way for all $i \leq L := \lceil \log_A r \rceil$. It is clear that $B_L = \{x\}$ which concludes to

$$\begin{aligned} \widetilde{p}_m(y, x) \geq & \min_{w_i \in \partial B_i} P_y(\tau_1 < m/2) \ldots \\ & \ldots P_{w_j}\left(\tau_j < \frac{m}{2^i}\right) ..P\left(\tau_L < \frac{m}{2^L}\right) \min_{0 \leq k \leq m-L} \widetilde{p}_k(x, x). \end{aligned}$$

From the initial conditions and (2.10) we have (5.25) for all $j$, so we can use a slight modification of Proposition 3.3 to get

$$P_w\left(\tau_i < \frac{m}{2^i}\right) > c_2$$

for all $w_j \in B(x, r_j)$ and $j$. Consequently, using (5.24) one has

$$\widetilde{p}_m(y, x) \geq \frac{c}{V(x, e(x, m))} c_2^L \geq \frac{c}{V(x, e(x, m))} r^{-D}$$

where $D = \frac{\log \frac{1}{c_2}}{\log A}$. ■

**Proof of Theorem 5.1.** The proof is a combination of two chaining. Let us recall that the time comparison principle implies (2.10), so the conditions of Proposition 5.3 are satisfied. First let us use Theorem 3.1 to reach the boundary of $B(x,r)$, where $r=\frac{d(x,y)}{3l-1}, l=l_9\left(x,y,\frac{n}{2},d(x,y)\right)$, then we use Proposition 5.3. The second inequality (5.23) follows from (5.22) if $n<c\frac{E(x,R)}{(\log E(x,R))^{\beta-1}}$ since in this case $\log r<Cl$ and it can be absorbed into the leading constant in the exponent. ■

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