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Random walk on graphs with regular resistance and volume growth

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Abstract. In this paper characterizations of graphs satisfying heat kernel estimates for a wide class of space–time scaling functions are given. The equivalence of the two-sided heat kernel estimate and the parabolic Harnack inequality is also shown via the equivalence of the upper (lower) heat kernel estimate to the parabolic mean value (and super mean value) inequality.

Résumé. Dans cet article, nous caractérisons des graphes qui satisfont des estimées du noyau de la chaleur pour un large ensemble de fonctions d'echelles spaciaux–temporelles. L'équivalence entre l'estimée du noyau de la chaleur et l'inégalité parabolique de Harnack est également démontrée par l'équivalence de l'estimée haute (basse) du noyau de la chaleur et l'inégalité parabolique de la valeur moyenne (et de la valeur moyenne supérieure).

MSC: 60J10; 60J45

Keywords: Random walk; Heat kernel; Parabolic inequalities

1. Introduction

The heat propagation through a medium is determined by its heat capacity and conductance of the media. For the Euclidian space this observation goes back to Einstein. The heat diffusion has been a subject of interest in the discrete and continuous case for several decades and the fundamental results go back to the classical works of Aronson [1], Davies [9], Fabes and Stroock [12], Grigor'yan [13], Moser [26,27], Li and Yau [25], Saloff-Coste [30], Varadhan [38]. All these works are confined to homogeneous spaces. The diffusion in these spaces is typically located within the distance $\sqrt{t}$, at time t, from the starting point. In other words, the time–space scaling is $(\mathit{time})^{1/2}$ or the space–time scaling is $(\mathit{distance})^2$. The inhomogeneous case attracts more and more attention of physicists and mathematicians since the 80-s. Geometric and algebraic conditions are relaxed and fractals enriched the topics. (For recent results see [4,5,7,8,14,21,29].)

Delmotte has shown in [10] (in the spirit of the results for manifolds by Saloff-Coste [30] and Grigor'yan [13]) for general graphs that the two-sided Gaussian heat kernel estimate

$$\frac{c\exp(-Cd(x,y)^2/n)}{V(x,\sqrt{n})} \le \widetilde{p}_n(x,y) \le \frac{C\exp(-cd(x,y)^2/n)}{V(x,\sqrt{n})} \tag{1.1}$$

is equivalent to the parabolic Harnack inequality (with $R = R^2$ scaling, see all the formal definitions below).

In the last two decades several works have been devoted to fractals and fractal like graphs. One of the particular features of these structures is that the walk (or process) admits the space–time scaling function R^β with an exponent $\beta > 2$. For the continuous case the equivalence of the two-sided heat kernel estimate and the parabolic Harnack inequality with $F(x,R)=R^{\beta}$ scaling has been shown in [21] (see also [2,4,5,19] and [22]). In the graph case, the equivalence to the two-sided sub-Gaussian estimate (for $\beta>1$)

$$\frac{c\exp[-C(d^{\beta}(x,y)/n)^{1/(\beta-1)}]}{V(x,n^{1/\beta})}\le \widetilde{p}_n(x,y)\le \frac{C\exp[-c(d^{\beta}(x,y)/n)^{1/(\beta-1)}]}{V(x,n^{1/\beta})} \tag{1.2}$$

and other conditions was shown in [16]. For a wider set of space–time scaling functions $F(x,R)=F(R)$ the corresponding results were obtained in [36].

Barlow, Coulhon and Grigor'yan investigated in [6] the long time behavior of the heat kernel on manifolds using volume growth conditions. Here a more detailed picture will be provided covering on- and off-diagonal estimates under volume growth and potential theoretic conditions.

Among others Hino and Ramírez [20], Norris [28] and Sturm [33] (see also references there) studied the heat diffusion in Dirichlet spaces. Their approach uses the intrinsic metric which recovers the classical Gaussian heat kernel estimate. For us the metric is a priori given and the space–time scaling might be different from the classical R^2. (For more comments about the difference of the two approaches with respect of fractals see comments in Section 3.2 of [20].)

The present paper is partly motivated by the works Li and Wang [24] and Sung [31]. We prove in the context of weighted graphs that for a wide set of scaling functions the heat kernel upper estimate is equivalent to the parabolic mean value inequality (among others this is shown in [24] for the R^2 scaling). We also show (inspired by [31] confined to the R^2 scaling) that some lower estimates are equivalent to the super mean value inequality. As a consequence, we prove that the conjunction of the parabolic mean value and super mean value inequality is equivalent to the two-sided heat kernel estimate and to the parabolic Harnack inequality, as well.

Recent studies successfully transfer results obtained in continuous setting to the discrete graph case and vice versa (cf. [5,16,17,23]). For instance in [5] the proof of the equivalence of the parabolic Harnack inequality and two-sided heat kernel estimate (for the R^{β} scaling) is given for measure metric Dirichlet spaces via the graph case where the equivalence is known (cf. [4]). We treat the graph case while we believe that all the arguments and results can be transferred and are valid for measure metric spaces equipped with a strongly local, regular symmetric Dirichlet form and with the corresponding diffusion process.

The aim of the present paper is to relax, as much as possible, the conditions imposed on the space–time scaling function. We will consider graphs for which the space–time scaling function $F(x,R)$ is not uniform in the center x. One may feel that such a generalization is formal. A very simple example shows the opposite (see [36]), for the constructed graph neither the volume $V(x,R)$ nor the space–time scaling function $F(x,R)$ is uniform in $x\in\Gamma$ but heat kernel estimates hold.

Fractafolds, defined by Strichartz [32], are the continuous counterparts of such structures and as it is mentioned above we expect that the presented results are transferable to continuous spaces and to fractafolds.

Before we can state our results we need some definitions.

1.1. Basic definitions

Let us consider a countable infinite connected graph Γ. A weight function $\mu_{x,y}=\mu_{y,x}>0$ is given on the edges $x\sim y$. This weight induces a measure $\mu(x)$

$$\mu(x)=\sum_{y\sim x}\mu_{x,y},\qquad \mu(A)=\sum_{y\in A}\mu(y)$$

on the vertex set $A\subset\Gamma$ and defines a reversible Markov chain $X_n\in\Gamma$, i.e. a random walk on the weighted graph (Γ,μ) with transition probabilities

$$P(x,y)=\frac{\mu_{x,y}}{\mu(x)},$$

$$P_n(x,y)=\mathbb{P}(X_n=y|X_0=x)\quad\text{and the corresponding kernel,}$$

$$p_n(x,y)=\frac{1}{\mu(y)}P_n(x,y).$$

Let us use the notation $\widetilde{p}_n = p_{n+1} + p_n$. The graph is equipped with the usual (shortest path length) graph distance $d(x, y)$ and open metric balls are defined for $x \in \Gamma$, $R > 0$ as $B(x, R) = \{y \in \Gamma \colon d(x, y) < R\}$. The μ-measure of balls is denoted by

$$V(x, R) = \mu\bigl(B(x, R)\bigr). \tag{1.3}$$

For a set $A \subset \Gamma$ the killed random walk is defined by the transition operator restricted to $c_0(A)$ (to the set of functions with support in A) and the corresponding transition probability and kernel are denoted by $P_n^A(x, y)$ and $p_k^A(x, y)$.

The (heat) kernel $p_n(x, y)$ is the fundamental solution of the discrete heat equation on (Γ, μ):

$$\partial_n u = \Delta u, \tag{1.4}$$

where $\partial_n u = u_{n+1} - u_n$ is the discrete differential operator with respect of the time and $\Delta = P - I$ is the Laplace operator on Γ.

Definition 1.1. *Throughout the paper we will assume that condition* (p_0) *holds, that is, there is a universal* $p_0 > 0$ *such that for all* $x, y \in \Gamma$, $x \sim y$

$$\frac{\mu_{x,y}}{\mu(x)} \geq p_0. \tag{1.5}$$

Definition 1.2. *The weighted graph has the volume doubling* (*VD*) *property* (*c.f.*, [18]) *if there is a constant* $D_V > 0$ *such that for all* $x \in \Gamma$ *and* $R > 0$

$$V(x, 2R) \leq D_V V(x, R). \tag{1.6}$$

Notation 1.1. *For convenience we introduce a short notation for the volume of the annulus*: $v(x, r, R) = V(x, R) - V(x, r)$ *for* $R > r > 0, x \in \Gamma$.

Definition 1.3. *Now let us consider the exit time*

$$T_{B(x,R)} = \min\bigl\{k \colon X_k \notin B(x, R)\bigr\}$$

from the ball $B(x, R)$ *and its mean value*

$$E_z(x, R) = \mathbb{E}(T_{B(x,R)} | X_0 = z)$$

and let us use the notation

$$E(x, R) = E_x(x, R).$$

Definition 1.4. *We will say that the weighted graph* (Γ, μ) *satisfies the time comparison principle* (*TC*) *if there is a constant* $C_T > 1$ *such that for all* $x \in \Gamma$ *and* $R > 0$, $y \in B(x, R)$

$$\frac{E(x, 2R)}{E(y, R)} \leq C_T. \tag{1.7}$$

Definition 1.5. *We will say that the weighted graph* (Γ, μ) *satisfies the weak time comparison principle* (*wTC*) *if there is a constant* $C > 1$ *such that for all* $x \in \Gamma$ *and* $R > 0$, $y \in B(x, R)$

$$\frac{E(x, R)}{E(y, R)} \leq C. \tag{1.8}$$

Notation 1.2. *For a set $A \subset \Gamma$ denote the closure by*

$$\overline{A} = \{y \in \Gamma\colon \text{ there is an } x \in A \text{ such that } x \sim y\},$$

the boundary $\partial A = \overline{A} \setminus A$ and $A^c = \Gamma \setminus A$.

Definition 1.6. *A function h is harmonic on a set $A \subset \Gamma$ if it is defined on $\overline{A}$ and*

$$Ph(x) = \sum_y P(x, y)h(y) = h(x)$$

for all $x \in A$.

Definition 1.7. *The weighted graph (Γ, μ) satisfies the elliptic Harnack inequality (H) if there is a $C > 0$ such that for all $x \in \Gamma$ and $R > 0$ and for all $u \geq 0$ harmonic functions on $B(x, 2R)$ the following inequality holds*

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u. \tag{1.9}$$

One can check easily that for any fixed R_0 for all $R < R_0$ the Harnack inequality follows from (p_0).

Definition 1.8. *We define W_0 to be the set of functions which are candidates to be a space–time scaling function. In particular $F \in W_0$ if $F : \Gamma \times \mathbb{N} \to \mathbb{R}$ and*

(1) *there are $\beta > 1, \beta' > 0, c_F, C_F > 0$ such that for all $R > r > 0, x \in \Gamma, y \in B(x, R)$*

$$c_F \left(\frac{R}{r}\right)^{\beta'} \leq \frac{F(x, R)}{F(y, r)} \leq C_F \left(\frac{R}{r}\right)^{\beta}, \tag{1.10}$$

(2) *there is a $c > 0$ such that for all $x \in \Gamma$, $R > 0$*

$$F(x, R) \geq cR^2, \tag{1.11}$$

(3) $$F(x, R + 1) \geq F(x, R) + 1 \tag{1.12}$$
for all $R \in \mathbb{N}$.

Finally $F \in W_1$ if $F \in W_0$ and $\beta' > 1$ holds as well.

Remark 1.1. *We have by (1.12) that the function $F(x, R)$ is strictly increasing from $\mathbb{N}$ to $\mathbb{R}$ in the second variable consequently it has the generalized inverse $f : \mathbb{N} \to \mathbb{N}$:*

$$f(x, n) = \min\{R \in \mathbb{N}\colon F(x, R) \geq n\}.$$

In the whole sequel $f(x, n)$ is reserved for this inverse.

The function sets $W_1 \subset W_0$ will play a particular role in the whole sequel.

Sometimes we will refer to the upper and lower estimate in (1.10) for $x = y$ as the doubling and the anti-doubling property and in general, jointly we refer to them as doubling or regularity properties.

Definition 1.9. *We say that $PH(F)$, the parabolic Harnack inequality holds for a function F if for the weighted graph (Γ, μ) there is a constant $C > 0$ such that for any $x \in \Gamma$, $R, k \geq 0$ and any solution $u \geq 0$ of the heat equation (1.4) on $\mathcal{D} = [k, k + F(x, R)] \times B(x, 2R)$ the following is true. On the smaller cylinders defined by*

$$\mathcal{D}^- = \left[k + \frac{1}{4}F(x, R), k + \frac{1}{2}F(x, R)\right] \times B(x, R) \quad \text{and} \quad \mathcal{D}^+ = \left[k + \frac{3}{4}F(x, R), k + F(x, R)\right] \times B(x, R)$$

and taking $(n_-, x_-) \in \mathcal{D}^-$, $(n_+, x_+) \in \mathcal{D}^+$,

$$d(x_-, x_+) \le n_+ - n_- \tag{1.13}$$

the inequality

$$u_{n_-}(x_-) \le C\widetilde{u}_{n_+}(x_+)$$

holds, where we use the $\widetilde{u}_n = u_n + u_{n+1}$.

Definition 1.10. *Let us define the "volume" of a space–time cylinder* $D = [n, m] \times B(x, R)$ *where* $m > n$, $R > 0$ *by*

$$\nu(D) = [m - n]V(x, R).$$

Definition 1.11. *We say that* $PMV_\delta(F)$ *(the strong form of) the parabolic mean value inequality with respect to a function* F *holds on* (Γ, μ) *if for fixed constants* $0 \le c_1 < c_2 < c_3 < c_4 \le c_5, 0 < \delta \le 1$ *there is a* $C > 1$ *such that for arbitrary* $x \in \Gamma$ *and* $R > 0$, *using the notations* $F = F(x, R)$, $B = B(x, R)$, $\mathcal{D} = [0, c_5F] \times B$, $\mathcal{D}^- = [c_1F, c_2F] \times B(x, \delta R)$, $\mathcal{D}^+ = [c_3F, c_4F] \times B(x, \delta R)$ *for any nonnegative Dirichlet sub-solution of the heat equation*

$$\Delta^B u \ge \partial_n u$$

on $\mathcal{D}$, *the inequality*

$$\max_{\mathcal{D}^+} u \le \frac{C}{\nu(\mathcal{D}^-)} \sum_{(i,y)\in\mathcal{D}^-} u_i(y)\mu(y) \tag{1.14}$$

holds.

Definition 1.12. *We will use* $PMV(F)$ *if* $PMV_\delta(F)$ *holds for* $\delta = 1$.

Definition 1.13. *We say that (the strong form of) the parabolic super mean value inequality* $PSMV(F)$ *holds on* (Γ, μ) *with respect to a function* F *if there is an* $0 < \varepsilon < 1$ *such that for any constants* $0 < c_1 < c_2 < c_3 < c_4 \le c_5$, $c_4 - c_1 < \varepsilon$, *there are* δ, $c > 0$ *such that for arbitrary* $x \in \Gamma$ *and* $R > 0$, *using the notations* $F = F(x, R)$, $B = B(x, R)$, $\mathcal{D} = [0, c_5F] \times B$, $\mathcal{D}^+ = [c_3F, c_4F] \times B(x, \delta R)$, $\mathcal{D}^- = [c_1F, c_2F] \times B(x, \delta R)$ *for any nonnegative Dirichlet super-solution of the heat equation*

$$\Delta^B u \le \partial_n u$$

on $\mathcal{D}$, *the inequality*

$$\min_{\mathcal{D}^+} \widetilde{u}_k \ge \frac{c}{\nu(\mathcal{D}^-)} \sum_{(i,y)\in\mathcal{D}^-} \widetilde{u}_i(y)\mu(y) \tag{1.15}$$

holds.

Definition 1.14. *We introduce for* $A \subset \Gamma$

$$G^A(y, z) = \sum_{k=0}^{\infty} P_k^A(y, z),$$

the local Green function, which is the Green function of the killed walk and the corresponding Green's kernel as

$$g^A(y, z) = \frac{1}{\mu(z)} G^A(y, z).$$

Definition 1.15. *The Green kernel may satisfy the following properties. There are* $c, C > 0$ *and a function* F *such that for all* $x \in \Gamma$, $R > 0$, $A = B(x, R) \setminus B(x, R/2)$, $B = B(x, 2R)$

$$\max_{y\in A} g^B(x, y) \le C\frac{F(x, 2R)}{V(x, 2R)}, \tag{1.16}$$

$$\min_{y\in A} g^B(x, y) \ge c\frac{F(x, 2R)}{V(x, 2R)}. \tag{1.17}$$

If both inequalities hold this fact will be denoted by $g(F)$.

1.2. Statement of the results

The main results of the paper are the following theorem.

Theorem 1.1. *If a weighted graph* (Γ, μ) *satisfies* (p_0) *and* (VD), *then the following statements are equivalent.*

(1) *There is an* $F \in W_0$ *such that* $g(F)$ *is satisfied,*
(2) (wTC) *and* (H) *hold,*
(3) *there is an* $F \in W_0$ *such that the upper estimate* $UE(F)$ *holds: there are* $C, \beta > 1, c > 0$ *such that for all* $x, y \in \Gamma$, $n > 0$

$$p_n(x, y) \le \frac{C}{V(x, f(x, n))}\exp\left[-c\left(\frac{F(x, d(x, y))}{n}\right)^{1/(\beta-1)}\right] \tag{1.18}$$

and furthermore the particular lower estimate $PLE(F)$ *holds: there are* $0 < c, \delta, \varepsilon < 1$ *such that for all* $x \in \Gamma$, $R > 0$, $B = B(x, R)$, $d(x, y) < n \wedge \delta f(x, n)$, $n \le \varepsilon F(x, R)$,

$$\widetilde{p}_n^B(x, y) \ge \frac{c}{V(x, f(x, n))}, \tag{1.19}$$

where $f(x, n)$ *is the inverse of* $F(x, R)$ *in the second variable,*
(4) *there is an* $F \in W_0$ *such that* $PMV(F)$ *and* $PSMV(F)$ *hold.*

Further equivalent conditions will be given in Section 4.

Remark 1.2. *The off-diagonal lower estimate* $LE(F)$ *which states that there are* $C, \beta' > 1, c > 0$ *such that for all* $x, y \in \Gamma$, $n \ge d(x, y)$

$$\widetilde{p}_n(x, y) \ge \frac{c}{V(x, f(x, n))}\exp\left[-C\left(\frac{F(x, d(x, y))}{n}\right)^{1/(\beta'-1)}\right] \tag{1.20}$$

can be obtained from (VD) *and* $PLE(F)$ *if* $\beta' > 1$ *in* (1.10) *using Aronson's classical chaining argument. This indicates the possibility to obtain two-sided heat kernel estimate and necessary and sufficient conditions for it.*

Theorem 1.2. *If a weighted graph* (Γ, μ) *satisfies* (p_0), *then the following statements are equivalent:*

(1) (VD) *holds and there is an* $F \in W_1$ *such that* $g(F)$ *is satisfied,*
(2) *there is an* $F \in W_1$ *such that the two-sided heat kernel estimate hold: there are* $C, \beta \ge \beta' > 1, c > 0$ *such that for all* $x, y \in \Gamma$, $n \ge d(x, y)$

$$c\frac{\exp[-C(F(x, d)/n)^{1/(\beta'-1)}]}{V(x, f(x, n))} \le \widetilde{p}_n(x, y) \le C\frac{\exp[-c(F(x, d)/n)^{1/(\beta-1)}]}{V(x, f(x, n))}, \tag{1.21}$$

where we write $d = d(x, y)$,

(3) *there is an* $F \in W_1$ *such that* $PMV(F)$ *and* $PSMV(F)$ *hold*,
(4) *there is an* $F \in W_1$ *such that* $PH(F)$ *holds*.

Let us remark that the observation that the pair of the upper and particular lower estimate is equivalent to the β-parabolic Harnack inequality goes back to [21]. It is also worth mentioning that the conditions (VD), the uniformity of the mean exit time and (H) are partially independent as the interesting examples given in [11] by Delmotte show.

Our results are presented in the discrete case, and with this limitation (which we consider not essential) are generalizations of several works devoted to heat kernel estimates and the parabolic Harnack inequality for scaling function $F(x, R) = R^2$, R^β or $F(R)$, among others [8,13,15–17,30,31].

The following elements of the present paper are new:

(1) the wide sets W_0 and W_1 of space–time scaling functions,
(2) condition $g(F)$ with respect to $F \in W_i$, $i = 0, 1$,
(3) the parabolic inequalities with respect to $F \in W_i$, $i = 0, 1$,
(4) the proof of the equivalence of the conjunction of the parabolic mean and super mean value inequality to the parabolic Harnack inequality,
(5) the role of the strong anti-doubling property, $\beta' > 1$ is explained.

The condition $g(F)$ is the generalization of the corresponding conditions (G) in [15] and (G_β) in [16].

In [31] partial equivalence (for $F(x, R) = R^2$) was shown for the parabolic super mean value inequality and the particular heat kernel lower estimate. Here we prove full equivalence for all $F \in W_0$ for a slightly modified version of the parabolic super mean value inequality. This modification allows us not only to show the full equivalence, but it is also appropriate for proving that the conjunction of the parabolic mean and super mean value inequality is equivalent to the parabolic Harnack inequality. We have not found such a result in the literature even for the classical case $F(x, R) = R^2$.

In most earlier works the space-homogeneous case $E(x, R) \simeq F(R)$ is considered. In these situations $\beta' > 1$ follows from the homogeneity (and from the other conditions needed for the heat kernel estimates).

The structure of the paper is the following. Section 2 contains the basic definitions. Section 3 recalls the results regarding the mean exit time. Section 4 contains the proof of Theorem 1.1. First the Section 4.1 summarizes a result about the heat kernel upper estimate, and Sections 4.2 and 4.3 the lower estimate. The proof of Theorem 1.2 which contains the parabolic Harnack inequality is given in Section 5. The paper is closed with a short remark on the homogeneous case.

2. Definitions and preliminaries

2.1. The volume

Definition 2.1. *We will use the inner product with respect to* μ:

$$(f, g)_\mu = \sum_{x \in \Gamma} f(x) g(x) \mu(x).$$

Remark 2.1. *One can show that* (VD) *is equivalent to*

$$\frac{V(x, R)}{V(y, S)} \le C \left(\frac{R}{S}\right)^\alpha,$$

where $\alpha = \log_2 D_V$ *and* $d(x, y) \le R$.

Remark 2.2. *It is easy to show* (*cf.* [8]) *that the volume doubling property implies the anti-doubling property*: *there is an* $A_V > 1$ *such that for all* $x \in \Gamma$, $R > 0$

$$2V(x, R) \le V(x, A_V R), \tag{2.1}$$

which is equivalent with the existence of $c, \alpha' > 0$ *such that for all* $x \in \Gamma$, $R > r > 0$

$$\frac{V(x,R)}{V(x,r)} \geq c\left(\frac{R}{r}\right)^{\alpha'}.$$

Notation 2.1. *For two real series* $a_\xi, b_\xi, \xi \in S$ *we shall use the notation* $a_\xi \simeq b_\xi$ *if there is a* $C > 1$ *such that for all* $\xi \in S$

$$C^{-1} a_\xi \leq b_\xi \leq C a_\xi.$$

Remark 2.3. *Another direct consequence of* (p_0) *and* (VD) *is that*

$$v(x,R,2R) = V(x,2R) - V(x,R) \simeq V(x,R) \tag{2.2}$$

2.2. Laplacian

Definition 2.2. *The random walk on the weighted graph is a reversible Markov chain and the Markov operator* P *is naturally defined by*

$$Pf(x) = \sum P(x,y) f(y).$$

Definition 2.3. *The Laplace operator on the weighted graph* (Γ, μ) *is defined simply as*

$$\Delta = P - I.$$

Definition 2.4. *The Laplace operator with Dirichlet boundary conditions on a finite set* $A \subset \Gamma$ *is defined as*

$$\Delta^A f(x) = \begin{cases} \Delta f(x) & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The smallest eigenvalue of $-\Delta^A$ *is denoted in general by* $\lambda(A)$ *and for* $A = B(x,R)$ *it is denoted by* $\lambda = \lambda(x,R) = \lambda(B(x,R))$.

Definition 2.5. *The energy or Dirichlet form* $\mathcal{E}(f,f)$ *associated to* (Γ, μ) *is defined as*

$$\mathcal{E}(f,f) = -(\Delta f, f)_\mu = \frac{1}{2} \sum_{x,y \in \Gamma} \mu_{x,y} \big(f(x) - f(y)\big)^2.$$

Using this notation the smallest eigenvalue of $-\Delta^A$ can be defined by

$$\lambda(A) = \inf\left\{\frac{\mathcal{E}(f,f)}{(f,f)_\mu}:\ f \in c_0(A),\ f \neq 0\right\} \tag{2.3}$$

as well.

2.3. The resistance

Definition 2.6. *For any two disjoint sets,* $A, B \subset \Gamma$, *the resistance,* $\rho(A,B)$, *is defined as*

$$\rho(A,B) = \big(\inf\big\{\mathcal{E}(f,f):\ f|_A = 1,\ f|_B = 0\big\}\big)^{-1}$$

and we introduce

$$\rho(x,r,R) = \rho\big(B(x,r), \Gamma \setminus B(x,R)\big)$$

for the resistance of the annulus around $x \in \Gamma$, *with* $R > r > 0$.

Definition 2.7. *We say that the product of the resistance and volume of the annulus is uniform in the space if*

$$\rho(x, R, 2R)v(x, R, 2R) \simeq \rho(y, R, 2R)v(y, R, 2R). \tag{2.4}$$

We will refer to this property shortly by (ρv).

Lemma 2.4. *For all weighted graphs*, $x \in \Gamma$, $R > r > 0$

$$\rho(x, r, R)v(x, r, R) \geq (R - r)^2. \tag{2.5}$$

For the proof see [35].

Definition 2.8. *The resistance lower estimate* $(RLE)(F)$ *holds for a function* F *if there is a* $c > 0$ *such that for all* $x \in \Gamma$, $R > 0$

$$\rho(x, R, 2R) \geq c\frac{F(x, 2R)}{V(x, 2R)}. \tag{2.6}$$

Definition 2.9. *The anti-doubling property* $(aD\rho v)$ *is satisfied for* ρv *if there are* $c, \beta' > 0$ *such that for all* $x \in \Gamma$, $R > r > 0$

$$\frac{\rho(x, R, 2R)v(x, R, 2R)}{\rho(x, r, 2r)v(x, r, 2r)} \geq c\left(\frac{R}{r}\right)^{\beta'}. \tag{2.7}$$

2.4. The mean exit time

Let us introduce the exit time T_A from a set $A \subset \Gamma$.

Definition 2.10. *The exit time from a set* A *is defined as*

$$T_A = \min\{k\colon\ X_k \in \Gamma \setminus A\},$$

its expected value is denoted by

$$E_x(A) = \mathbb{E}(T_A | X_0 = x)$$

and furthermore let us us write

$$\overline{E}(x, R) = \max_{y \in B(x,R)} E_y\big(B(x, R)\big).$$

In this section we introduce some properties of the mean exit time which will play a crucial role in the whole sequel. First of all it is immediate that

$$E(x, 1) \geq 1$$

and for $R \in \mathbb{N}$

$$E(x, R + 1) \geq E(x, R) + 1. \tag{2.8}$$

Remark 2.5. *We have by* (2.8) *that the function* $E(x, R)$ *is strictly increasing from* $\mathbb{N}$ *to* $\mathbb{R}$ *in the second variable consequently it has the generalized inverse* $e : \mathbb{N} \to \mathbb{N}$:

$$e(x, n) = \min\big\{R \in \mathbb{N}\colon\ E(x, R) \geq n\big\}.$$

Remark 2.6. *It is easy to see that* (*TC*) *is equivalent to the existence of constants* C, $\beta \geq 1$ *for which*

$$\frac{E(x,R)}{E(y,S)} \leq C\left(\frac{R}{S}\right)^{\beta}, \tag{2.9}$$

for all $y \in B(x,R)$, $R \geq S > 0$.

Definition 2.11. *The local sub-Gaussian upper exponent, with respect to a function* $F(x,R)$ *is* $k = k_x(n,R) \geq 1$, *it is defined as the maximal integer for which*

$$\frac{n}{k} \leq q \min_{y \in B(x,R)} F\left(y, \left\lfloor \frac{R}{k} \right\rfloor\right) \tag{2.10}$$

or $k = 1$ *by definition if there is no appropriate* k. *Here* $q > 0$ *is a small fixed constant* (*cf.* [34], $q < \min\{1/16, c_F p_0/C_F\}$).

Definition 2.12. *Let* $n \geq l_x = l_x(n,R) \geq 1$ *be the minimal integer for which*

$$\frac{n}{l} \geq CF\left(x, \left\lceil \frac{R}{l} \right\rceil\right) \tag{2.11}$$

or $l = n$ *by definition if there is no appropriate* l. *The constant* C *will be specified later.*

Definition 2.13. *The local sub-Gaussian lower exponent* $l(n,R,A)$ *with respect to a function* $F(x,R)$ *for* $A \subset \Gamma$ *is the maximal integer* l *for which*

$$\frac{n}{l} \geq C \max_{z \in A} F\left(z, \left\lceil \frac{R}{l} \right\rceil\right). \tag{2.12}$$

Definition 2.14. *The global sub-Gaussian exponent* $m = m(n,R)$ *is defined as the maximal integer for which*

$$\frac{n}{m} \leq q \min_{y \in \Gamma} E\left(y, \left\lfloor \frac{R}{m} \right\rfloor\right) \tag{2.13}$$

or $m = 1$ *by definition if there is no appropriate* m.

Definition 2.15. *The mean exit time is uniform in the space if there is a function* F *such that*

$$E(x,R) \simeq F(R). \tag{2.14}$$

This property will be referred to by (E).

The definition $m(n,R)$ is prepared for the particular case when $E(y,R) \simeq E(x,R) \simeq F(R)$, i.e. E is basically independent of $x, y \in \Gamma$.

Definition 2.16. *We define* V_1 *to be the a set of functions such that* $F \in V_0$ *if* $F : \Gamma \times \mathbb{N} \to \mathbb{R}$ *and there are* $\beta' > 1$, $c_F > 0$ *such that for all* $R > r > 0$, $x \in \Gamma$, $y \in B(x,R)$

$$c_F\left(\frac{R}{r}\right)^{\beta'} \leq \frac{F(x,R)}{F(y,r)}. \tag{2.15}$$

2.5. Mean value inequalities

Definition 2.17. *The elliptic mean value inequality* (*MV*) *holds if there is a* $C > 0$ *such that for all* $x \in \Gamma$, $R > 0$ *and for all* $u \geq 0$ *harmonic functions on* $B = B(x, R)$

$$u(x) \leq \frac{C}{V(x, R)} \sum_{y \in B} u(y)\mu(y), \tag{2.16}$$

Remark 2.7. *Let us recognize that in the definition of PMV* δ *is a "free" parameter, while in the definition of* $PSMV(F)$ *it depends on* ε *and* $c_i, i = 1, \ldots, 5$. *Let us also observe that* c_i*s are subject of the restriction* $c_4 - c_1 < \varepsilon$.

Remark 2.8. *One should note that the definition of the parabolic mean value inequality is slightly different from the one given in* [36]. *There it is stated for Dirichlet solutions, here we have it for arbitrary Dirichlet sub-solutions. It is easy to see that the extended definitions fit into Theorem* 4.2. *On one hand solutions are sub-solutions, on the other hand, the proof of the implication* $UE \Rightarrow PMV$ *of Theorem* 4.2 *follows word by word for sub-solutions.*

Remark 2.9. *The condition* (1.13) *in the definition of the parabolic Harnack inequality is needed in order to have a path* (*with nonzero probability*) *of length no more than* $n_+ - n_-$ *between* x_- *and* x_+. *One can eliminate this restriction if the parabolic Harnack inequality is considered only for large enough* R*s. The condition* $n_+ - n_- \geq d(x_-, x_+)$ *is satisfied if* $(c_3 - c_2)F(x, R) \geq cR^2 > 4R$ *which holds if* $R > R_0$. *Such an* R_0 *depends only on the constants* (*cf.* [35]). *In order to avoid lengthy technical discussion we may assume* $R > R_0$ *in all these situations. If not otherwise stated, the corresponding inequalities for* $R \leq R_0$ *follow from* (p_0).

3. Properties of the mean exit time

In this section we recall some results from [35] which describe the behavior of the mean exit time. If this is not mentioned otherwise, the statements and proofs can be found in [35]. The first one is the Einstein relation.

Theorem 3.1. *If* (p_0), (*VD*), (H) *and one of the conditions* (*wTC*), $(aD\rho v)$, $RLE(E)$ *or* $\rho v \in W_0$ *hold, then* (*ER*), *the Einstein relation*

$$E(x, 2R) \simeq \rho(x, R, 2R)v(x, R, 2R) \tag{3.1}$$

holds, furthermore

$$\lambda^{-1}(x, R) \simeq E(x, R) \simeq \overline{E}(x, R)$$

and

$$E(x, R) \in W_0.$$

Theorem 3.2. *If* (p_0), (*VD*), (*TC*) *hold, then the Einstein relation*

$$E(x, 2R) \simeq \rho(x, R, 2R)v(x, R, 2R) \tag{3.2}$$

holds, furthermore

$$\lambda^{-1}(x, R) \simeq \overline{E}(x, R) \simeq E(x, R) \in W_0.$$

The properties of the inverse function e and properties of E are linked as the following evident lemma states.

Lemma 3.1. *The following statements are equivalent*:

1. *There are* $C, c > 0$, $\beta \geq \beta' > 0$ *such that for all* $x \in \Gamma$, $R \geq r > 0$, $y \in B(x, R)$

$$c\left(\frac{R}{r}\right)^{\beta'} \leq \frac{E(x,R)}{E(y,r)} \leq C\left(\frac{R}{r}\right)^{\beta}. \tag{3.3}$$

2. *There are* $C, c > 0$, $\beta \geq \beta' > 0$ *such that for all* $x \in \Gamma$, $n \geq m > 0$, $y \in B(x, e(x,n))$

$$c\left(\frac{n}{m}\right)^{1/\beta} \leq \frac{e(x,n)}{e(y,m)} \leq C\left(\frac{n}{m}\right)^{1/\beta'}. \tag{3.4}$$

Lemma 3.2. *If* $F \in W_0$, *then for* $k_x(n, R)$ *defined in* (2.10)

$$k_x(n,R) + 1 \geq c\left(\frac{F(x,R)}{n}\right)^{1/(\beta-1)} \tag{3.5}$$

for all $x \in \Gamma$, $R, n > 0$ *for fixed* $c > 0$, $\beta > 1$. *In addition, if* $F \in W_1$ *is assumed, then*

$$l_x(n,R) - 1 \leq C\left(\frac{F(x,R)}{n}\right)^{1/(\beta'-1)}. \tag{3.6}$$

Proof. The statement follows from the regularity properties of F easily, $\beta > 1$ is ensured by $\beta \geq 2$ (see (1.11)) and $\beta' > 1$ by the assumption. □

Lemma 3.3. *If* (E) *is satisfied and* $E \in W_0$ *then*

$$k_x(n,R) \simeq l_x(n,R) \simeq m(n,R).$$

Since this fact is not used in the proof of the main results, the elementary proof is omitted (for some hints see [34]).

Remark 3.4. *Let us mention here that under* (p_0), (VD) *and* (H) *the uniformity of the mean exit time in the space*:

$$E(x,R) \simeq F(R)$$

ensures that E *satisfies the left-hand side of* (1.10) *with a* $\beta' > 1$ (*cf.* [35]). *This explains that in the "classical" cases, when* (E) *holds one should not assume* $\beta' > 1$, *it follows from the conditions* (*see also* [36], *Theorem* 4.13).

Lemma 3.5. *For* (Γ, μ) *for all* $x \in \Gamma$, $R > 0$

$$\min_{z \in \partial B(x,(3/2)R)} E\left(z, \frac{R}{2}\right) \leq \rho(x,R,2R)v(x,R,2R). \tag{3.7}$$

Corollary 3.6. *Under* (p_0) *and* (VD)

$$\exists F \in W_0: \quad g(F) \quad \Longleftrightarrow \quad (H) + (ER).$$

Proof. The implication $\Leftarrow$ was shown in [35]. We also know that (p_0), (VD), (H) and (ER) implies (TC) hence by Theorem 3.2 $E \in W_0$. The reverse implication needs some additional arguments. We know again from [35] that $g(F)$ implies (H). We show here the implication $g(F) \Rightarrow (ER)$ under (p_0), (VD) and (H). That needs some care. Let us assume that $r_i = 2^i$, $r_{n-1} < 2R \leq r_n$, $B_i = B(x, r_i)$, $A_i = B_i \setminus B_{i-1}$, $V_i = V(x, r_i)$. In [16] Section 4.3 it is derived using (p_0), (VD) and (H) that

$$E(x,2R) \leq C\sum_{i=0}^{n-1} V_{i+1}\rho(x,r_i,r_{i+1}).$$

Now we use a consequence of (H):

$$\rho(x,r_i,r_{i+1}) \le C \max_{y\in A_{i+1}} g^{B_{i+1}}(x,y)$$

(see for instance [35], Section 4. or [3]) to obtain

$$\begin{aligned} E(x,2R) &\le C\sum_{i=0}^{n-1} V_{i+1} \max_{y\in A_{i+1}} g^{B_{i+1}}(x,y) \\ &\le C\sum_{i=0}^{n-1} F(x,r_{i+1}) \le CF(x,r_n)\sum_{i=0}^{n-1} 2^{-i\beta'} \\ &\le CF(x,2R), \end{aligned}$$

where (1.16) was used to get the second inequality.

On the other hand, from (1.17) one obtains

$$\begin{aligned} c\frac{F(x,2R)}{V(x,2R)}\left(V(x,R)-V\left(x,\frac{R}{2}\right)\right) &\le \min_{y\in B(x,R)\backslash B(x,R/2)} g^B(x,y) \sum_{z\in B(x,R)\backslash B(x,R/2)} \mu(z) \\ &\le \sum_{z\in B(x,R)\backslash B(x,R/2)} g^B(x,z)\mu(z) \le E(x,2R), \end{aligned}$$

this means that

$$cF(x,2R) \le E(x,2R)$$

consequently, $F \simeq E$, $E \in W_0$ and (TC) is satisfied. Finally by Theorem 3.2 the conditions (p_0), (VD) and (TC) imply (ER). □

Corollary 3.7. *Assume that (Γ,μ) satisfies (p_0), (VD) and (H), then*

$$\begin{aligned} (wTC) &\iff (aD\rho v) \iff (TC) \iff (ER) \iff RLE(E) \\ &\overset{(i)}{\iff} \text{there is an } F\in W_0 \text{ such that } g(F). \end{aligned} \tag{3.8}$$

Proof. Except the last implications the statement was shown in [35] while the last one is just Corollary 3.6. □

Remark 3.8. *Let us remark here that as a side result it follows that $RLE(E)$ or $g(F)$ for $F\in W_0$ implies $\rho v \simeq F$ and $E \simeq F$ as well.*

4. Temporal regularity and heat kernel estimates

In this section we prove Theorem 1.1 in an extended form. We have seen in Corollary 3.7 that under the conditions (p_0), (VD) and (H)

$$(wTC) \iff (aD\rho v) \iff (TC) \iff (ER) \iff RLE(E). \tag{4.1}$$

Let $(*)$ denote any of the equivalent conditions. Using this convention we can state the extension of Theorem 1.1 as follows.

Theorem 4.1. *If a weighted graph (Γ,μ) satisfies (p_0) and (VD), then the following statements are equivalent*:

(1) *there is an* $F \in W_0$ *such that* $g(F)$ *is satisfied*,
(2) (H) *and* $(*)$ *hold*,
(3) *there is an* $F \in W_0$ *such that* $UE(F)$ *and* $PLE(F)$ *are satisfied*,
(4) *there is an* $F \in W_0$ *such that* $PMV(F)$ *and* $PSMV(F)$ *are satisfied.*

The proof of Theorem 4.1 contains two autonomous results. The first one states that the upper estimate is equivalent to the parabolic mean value inequality, the second one states that the particular lower estimate is equivalent to the parabolic super mean value inequality. The return route from (5) to (1) and (2) is based on the Einstein relation, on a potential theoretic result from [35] and a modification of the return route developed in [34]. The proof of (2) ⇒ (3) generalizes methods of [15] and [16].

Let us emphasize the importance of the condition $(aD\rho v)$ in (4.1). It is a condition on the volume and resistance; no assumption of stochastic nature is involved so the result is in the spirit of Einstein's observation on the heat propagation. These conditions in conjunction with (VD) and (H) provide the characterization of the heat kernel estimates in terms of volume and resistance properties. Of course the elliptic Harnack inequality is not easy to verify. Meanwhile we learn from $g(F)$ that the main properties ensured by the elliptic Harnack inequality are that the equipotential surfaces of the local Green kernel $g^{B(x,R)}$ are basically spherical and the potential growth is regular (cf. [35]).

4.1. The upper estimate

This section provides the upper bound part of the implication (2) ⇒ (3) of Theorem 1.1. In detail

$$\left.\begin{array}{c}(VD)\\(TC)\\(H)\end{array}\right\} \quad\Longrightarrow\quad DUE(E) \tag{4.2}$$

and under (VD) and (TC)

$$DUE(E) \quad\Longleftrightarrow\quad UE(E) \quad\Longleftrightarrow\quad PMV(E).$$

In particular the parabolic mean value inequality is shown to be equivalent to the upper estimate and the other conditions. This result has been proved in [36]:

Theorem 4.2. *For a weighted graph* (Γ,μ) *if* (p_0), (VD), (TC) *conditions hold*, *then the following statements are equivalent*:

(1) *the local diagonal upper estimate* $DUE(E)$ *holds*; *there is a* $C>0$ *such that for all* $x\in\Gamma$, $n>0$

$$p_n(x,x)\le\frac{C}{V(x,e(x,n))}, \tag{4.3}$$

(2) *the upper estimate* $UE(E)$ *holds*: *there are* $C,\beta>1$, $c>0$ *such that for all* $x,y\in\Gamma$, $n>0$

$$p_n(x,y)\le\frac{C}{V(x,e(x,n))}\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{1/(\beta-1)}\right],$$

(3) *the parabolic mean value inequality*, $PMV(E)$ *holds*,
(4) *the mean value inequality*, (MV) *holds.*

Corollary 4.1. *If* (Γ,μ) *satisfies* (p_0), *then*

$$\left.\begin{array}{c}(VD)\\(wTC)\\(H)\end{array}\right\} \quad\Longrightarrow\quad UE(E).$$

Proof. We know from Theorem 3.2 that $E \in W_0$. The statement follows from Corollary 3.7 and Theorem 4.2 since the elliptic Harnack inequality, (H) implies the elliptic mean value inequality (MV). $\square$

Remark 4.2. *The equivalences in* (4.2) *for* $F \in W_0$ *instead of* E *follow from the same proofs given in* [36] *for* E (*see also Corollary* 3.7 *and Remark* 3.8).

Remark 4.3. *Consequently the implication of the upper bound part in Theorem* 4.1 (*and Theorem* 1.1) (2) $\Rightarrow$ (3) *is shown*.

Remark 4.4. *The implication for an* $F \in W_0$

$$DUE(F) \quad \Longleftrightarrow \quad UE(F) \quad \Longrightarrow \quad PMV(F) \tag{4.4}$$

in particular $DUE(F) \Rightarrow PMV(F)$ *can be shown repeating step by step the proof given for the particular space–time scaling function* E. *The full proof is spelled out in* [37], *Theorem* 8.6. *This gives the proof of the upper bound part of the implication of Theorem* 4.1 (3) $\Rightarrow$ (4). *Let us note that the conditions needed to deduce the implication in* (4.4) *are only* (p_0), (VD) *and* $F \in W_0$.

4.2. The near diagonal lower estimate

In this section we give a lower estimate for the Dirichlet heat kernel and for the global one.

Definition 4.1. *The near diagonal lower estimate*, $NDLE(F)$ *holds with respect to a function* F *if there are* $c, \delta > 0$ *such that for all* $x, y \in \Gamma$, $n > 0$, $d(x, y) < \delta f(x, n) \wedge n$

$$\widetilde{p}_n(x, y) \ge \frac{c}{V(x, f(x, n))}, \tag{4.5}$$

where f *is the existing inverse of* F *in the second variable*, *defined in Remark* 1.1.

Remark 4.5. *It is clear that* $PLE(F)$ *implies* $NDLE(F)$. *It is also known* (*cf.* [14]) *that* $NDLE(F)$ *and* $UE(F)$*implies* $PLE(F)$ *if* $F = R^2$, *the same proof works* $F \in W_0$.

Theorem 4.3. *For weighted graphs*

$$(p_0) + (VD) + (TC) + (H) \quad \Longrightarrow \quad PLE(E).$$

The proof closely follows the steps of the corresponding proof given for the case $E(x, R) \simeq R^\beta$ in [15] therefore it is omitted.

Remark 4.6. *From the regularity of* V *and* F (*and* f) *it is immediate that* $PLE(F)$ *is equivalent with the slightly stronger form* $\delta' = \delta/2$: *for all* $x \in \Gamma$, $R > 0$, $B = B(x, R)$, $n \le \varepsilon F(x, R)$, $y, z \in B(x, \delta' f(x, n))$

$$\widetilde{p}_n^B(y, z) \ge \frac{c}{V(y, f(y, n))}. \tag{4.6}$$

4.3. The parabolic super mean value inequality

In this section the equivalence of the particular lower estimate and a kind of converse of the parabolic mean value inequality is shown. The partial equivalence for the classical ($F(x, R) = R^2$) and continuous situation was shown in [31]. Here the generalization to the present settings is provided.

For technical reasons we use some specific constants, like ε, δ from PLE, c_F, C_F from the definition of the set of scaling functions W_0 (in (1.10)).

In this section we show that for an $F \in W_0$

$$PLE(F) \iff PSMV(F),$$

that is, the following theorem holds.

Theorem 4.4. *For the weighted graph (Γ, μ) assume (p_0) and (VD). Then for an $F \in W_0$ $PLE(F)$ holds if and only if $PSMV(F)$ holds as well.*

Proof. The proof follows the main steps of [31]. We know that $PLE(F)$ implies the slightly stronger version (4.6) that is there are $\delta, \varepsilon > 0$ and $c > 0$

$$p_m^B(y,z) + p_{m+1}^B(y,z) \geq \frac{c}{V(y, f(y,m))} \tag{4.7}$$

holds, provided that $y, z \in B(x,r)$, where $r = \delta f(x,m)/2$, and $m \leq \varepsilon F = \varepsilon F(x,R)$. For the super-solution u we have that for all $c_3 F \leq k \leq c_4 F$, $c_1 F \leq i \leq c_2 F$

$$\widetilde{u}_k(y) \geq \sum_{z \in B(x,R)} \widetilde{p}_{k-i}^B(y,z)\mu(z)u_i(z).$$

In order to use (4.6), we choose

$$\delta^* = \frac{1}{C_F}(c_3 - c_2)^{1/\beta'}\delta/2, \tag{4.8}$$

which ensures that $y, z \in B(x,r)$ if $r = \delta^* R$. From the condition $c_4 - c_1 \leq \varepsilon$ it follows that $k - i \leq \varepsilon F(x,R)$ is satisfied and $PLE(F)$ can be applied:

$$\widetilde{u}_k(y) \geq \sum_{y \in B(x,\delta^* R)} \widetilde{p}_{k-i}^B(y,z)\mu(z)u_i(z) \geq \frac{c}{V(x, f(x,k-i))} \sum_{y \in B(x,\delta^* R)} \mu(y)u_i(y).$$

Now let us sum for $c_1 F \leq i \leq c_2 F$ and divide by $(c_2 - c_1)F$ to obtain

$$\widetilde{u}_k(y) \geq \frac{c}{F(x,R)} \sum_{i=c_1 F}^{c_2 F} \frac{1}{V(x, f(x,k))} \sum_{y \in B(x,\delta^* R)} \mu(y)u_i(z) \geq \frac{c}{V(x,R)F(x,R)} \sum_{i=c_1 F}^{c_2 F} \sum_{y \in B(x,\delta^* R)} \mu(y)u_i(z).$$

Now we prove the reverse implication $PSMV(F) \Rightarrow PLE(F)$ by applying $PSMV$ twice.

1. Denote ε, δ, c_i the constants in $PSMV(F)$, c_i will be specified later (which determines δ as well), furthermore $F = F(p,R)$, $B = B(p,R)$, $r_1 = R/8$, $F_1 = F(p,r_1)$, $D_1 = B(p,\delta r_1)$, $m = c' F_1$, $c_1 \leq c' \leq c_2$ and $D = B(p, \delta \frac{R}{4})$. Let us define

   $$u_n(y) = \begin{cases} \sum_{z \in D} \widetilde{p}_{n-m}^B(y,z)\mu(z) & \text{if } n > m, \\ 1 & \text{if } n \leq m. \end{cases}$$

   This is a solution on $D \times [0, \infty]$ of

   $$P^B u_n = u_{n+1}$$

   and $u_n \geq 0$. From the $PSMV(F)$ it follows that

   $$u_k(x) \geq \frac{c}{V(x,\delta r_1)F_1} \sum_{i=c_1 F_1}^{c_2 F_1} \sum_{w \in D_1} \mu(w)\widetilde{u}_i(w)$$

provided that, $x \in D_1$ and $c_3F_1 < k < c_4F_1$. From the definition of u_n, (VD) and $F \in W_0$ it follows that

$$u_k(x) \ge \frac{c}{V(x,\delta r_1)F_1} \sum_{i=c_1F_1}^{c_2F_1} \sum_{w\in D_1} c\mu(w) \ge c.$$

Again from the definition of u_n and $D_1 \subset D$, $c_3F_1 < k < c_4F_1$, $x \in D_1$ we obtain

$$\sum_{z\in D} \widetilde{p}^B_{k-m}(x,z)\mu(z) \ge c \tag{4.9}$$

or equivalently

$$\sum_{z\in D} \widetilde{p}^B_{i}(x,z)\mu(z) \ge c \tag{4.10}$$

if $x \in D_1$, $(c_3 - c_2)F_1 < i < (c_4 - c_1)F_1$.

2. We will use the parabolic super mean value inequality in a new ball B_2 for $\widetilde{p}^B_l(x,y)$ with the same set of constants c_i, hence with the same δ as well. Let $r_2 = R/2$, $B_2 = B(x,r_2)$, $D_2 = B(x,\delta r_2)$, $F_2 = F(x,r_2)$. We apply *PSMV* in B_2 and obtain that for $c_3F_2 < l < c_4F_2$, $y \in D_2$

$$\widetilde{p}^B_l(x,y) \ge \frac{c}{V(x,\delta r_2)F(x,r_2)} \sum_{i=c_1F_2}^{c_2F_2} \sum_{z\in D_2} \widetilde{p}^B_i(x,z)\mu(z)$$

if in addition $B_2 \subset B(p,R)$. Let $x \in B(p,\frac{\delta R}{8})$. This ensures that $B_2 \subset B(p,R)$ and $D_2 \supset D$ and we obtain for $y \in B(x,\frac{\delta R}{4}) \subset D_2$ (and $B(x,\frac{\delta R}{4}) \subset B(p,\frac{\delta R}{2})$ as well) that

$$\widetilde{p}^B_l(x,y) \ge \frac{c}{V(x,\delta r_2)F(x,r_2)} \sum_{c_1F_2}^{c_2F_2} \sum_{z\in D_2} \widetilde{p}_i{}^B(y,z)\mu(z). \tag{4.11}$$

In order to use (4.10) we require

$$c_2F_2 \ge (c_4 - c_1)F_1 \tag{4.12}$$

and

$$c_1F_2 \le (c_3 - c_2)F_1. \tag{4.13}$$

From the assumption $F \in W_0$ it follows that (4.12) is satisfied if

$$c_4 = c_2\bigl(1 + c_F 4^{\beta'}\bigr)$$

and (4.13) is satisfied if

$$c_1 = q\frac{(c_3 - c_2)}{C_F}4^{-\beta}$$

for any $0 < q < 1$. Finally

$$0 < c_4 - c_1 = c_2\bigl(1 + c_F 4^{\beta'}\bigr) - q\frac{(c_3 - c_2)}{C_F}4^{-\beta} < \varepsilon$$

and $c_1 < c_2 < c_3 < c_4$ can be ensured with the appropriate choice of c_2, c_3 and q. Using (4.12) and (4.13) and $D_2 \supset D$ the estimate in (4.11) can be continued as follows:

$$\widetilde{p}_l^B(x,y) \geq \frac{c}{V(x,\delta r_2)F(x,r_2)} \sum_{c_1F_2}^{c_2F_2} \sum_{z\in D_2} \widetilde{p}_i^{\,B}(y,z)\mu(z).$$

$$\geq \frac{c}{V(x,\delta r_2)F(x,r_2)} \sum_{(c_3-c_2)F_1}^{(c_4-c_1)F_1} \sum_{z\in D} \widetilde{p}_i^B(y,z)\mu(z).$$

Now we apply (4.10) to conclude to

$$\widetilde{p}_l^{B(p,R)}(x,y) \geq \frac{c}{V(x,\delta r_2)F(x,r_2)} \sum_{(c_3-c_2)F_1}^{(c_4-c_1)F_1} c \geq \frac{c}{V(x,R)} \geq \frac{c}{V(x,f(x,l))},$$

where $c_3F_2 \leq l \leq c_4F_2$ and $y \in B(p, \frac{\delta R}{4})$. Finally let $S \geq 2R$

$$\widetilde{p}_l^{B(p,S)}(x,y) \geq \widetilde{p}_l^{B(x,R)}(x,y) \geq \frac{c}{V(x,f(x,l))} \tag{4.14}$$

under the same conditions. Now choosing $\varepsilon' = \frac{c_F}{C_F}2^{-\beta}$ and $\delta' = \frac{\delta}{4}(c_3c_f)^{1/\beta'}$ (4.14) implies that $PLE(F)$ (in the stronger form: (4.6))

$$\widetilde{p}_l^{B(p,S)}(x,y) \geq \frac{c}{V(x,f(x,l))} \tag{4.15}$$

for $d(x,y) \leq \delta' f(p,l)$, $l \leq \varepsilon' F(p,S)$.

□

4.4. *Time comparison*

In this subsection we summarize the results which lead to the proof of (1) ⇔ (2) ⇒ (3) ⇒ (4) in Theorem 4.1 and we prove the return route from (4) ⇒ (2) The equivalence of (1) and (2) is established by Theorem 3.2, Corollaries 3.6 and 3.7, see also Remark 3.8. The implication (2) ⇒ (3) is given by Theorem 4.2 and 4.3 and (3) ⇒ (4) is a combination of the result stated in Remark 4.4 and in Theorem 4.4.

Now we prove (4) ⇒ (2), the return route of Theorem 4.1. Our task is to verify the implications in the diagram below under the assumption $F \in W_0$ and (p_0), (VD).

$$\left.\begin{matrix} PMV_1(F) \\ PSMV(F) \end{matrix}\right\} \implies \left.\begin{matrix} PMV_{\delta^*}(F) \\ PSMV(F) \end{matrix}\right\} \implies (H), \tag{4.16}$$

$$\left.\begin{matrix} PMV_1(F) \\ PSMV(F) \end{matrix}\right\} \implies \left.\begin{matrix} DUE(F) \\ PLE(F) \\ (H) \end{matrix}\right\} \implies \left.\begin{matrix} \rho v \simeq F \\ (H) \end{matrix}\right\} \implies \begin{matrix} (TC) \\ (H) \end{matrix}. \tag{4.17}$$

The heat kernel estimates are established as we indicated above. Now we deal with proof of the elliptic Harnack inequality (H) and the time comparison principle (TC).

Theorem 4.5. *If Γ satisfies (p_0), (VD) and there is an $F \in W_0$ for which $PMV(F)$ and $PSMV(F)$ are satisfied, then the elliptic Harnack inequality holds on (H).*

We need an intermediate step, the parabolic mean value inequality for smaller balls. We choose a particular set of constants c_i, subject to some restrictions coming from $PSMV$ and needed for later use.

Lemma 4.7. *If (Γ, μ) satisfies (p_0), (VD) and $PMV_1(F)$ for an $F \in W_0$, then for a given $\varepsilon, \delta > 0$, $0 < \delta^* \leq \frac{1}{C_F}\varepsilon^{1/\beta'}\frac{\delta}{2}$ there are $c_1 < \cdots < c_4$ such that $PMV_{\delta^*}(F)$ holds for ε and $c_i - s$.*

Proof. We would like to derive $PMV_{\delta^*}(F)$ for c_i from $PMV_1(F)$ which holds for some other constants a_i. We will apply $PMV_1(F)$ on the ball $B = B(x, \delta R)$ and re-scale the time accordingly. We have $PMV_{\delta^*}(F)$ on $B(x, R)$ by

$$\begin{aligned}
\max_{\substack{c_3 F(x,R) \leq i \leq c_4 F(x,R),\\ y \in B}} u_i(y) &\leq \max_{\substack{a_3 F(x,\delta R) \leq i \leq a_4 F(x,\delta R),\\ y \in B}} u_i(y)\\
&\leq \frac{C}{\nu(D^-)} \sum_{j=a_1 F(x,\delta R)}^{a_2 F(x,\delta R)} \sum_{y \in B} u_j(z)\mu(z)\\
&\leq \frac{C}{\nu(D^-)} \sum_{j=c_1 F(x,R)}^{c_2 F(x,R)} \sum_{y \in B} u_j(z)\mu(z)
\end{aligned}$$

if the inequalities $c_1 < \cdots < c_4, a_1 < \cdots < a_4$,

$$\begin{aligned}
&a_4 F(x, \delta R) \geq c_4 F(x, R),\\
&a_3 F(x, \delta R) \leq c_3 F(x, R),\\
&a_2 F(x, \delta R) \leq c_2 F(x, R),\\
&a_1 F(x, \delta R) \geq c_1 F(x, R).
\end{aligned}$$

are satisfied. We require in addition that $c_4 \leq \varepsilon$ and $\frac{1}{C_F}(c_3 - c_2)^{1/\beta'}\frac{\delta}{2} \geq \delta^*$. One can see that the following choice satisfies these restrictions. Denote $p = C_F(\delta^*)^{\beta}$, $q = c_F(\delta^*)^{\beta'}$. Let

$$\begin{aligned}
c_4 &= \varepsilon, & a_4 &= \frac{2q}{p} c_4,\\
c_3 &< c_4, & a_3 &= q c_3,\\
c_2 &< c_3, & a_2 &= \frac{1}{2}\min\{p c_2, a_3\},\\
c_1 &= \frac{1}{2}\min\left\{\frac{a_2}{q}, c_2\right\}, & a_1 &= q c_1.
\end{aligned}$$

Let us observe that c_1 can be arbitrarily small since $c_4 \leq \varepsilon$ and if the sub-solution is not given from an m up to $a_4 F(x, \delta R)$ it can be extended simply by $u_{i+m} = P_i^{B(x,R)} u_m$. □

Proof of Theorem 4.5. Let us fix a set of constants $c_1 < c_2 < c_3 < c_4 = \varepsilon$ as in Lemma 4.7 and apply $PSMV(F)$ for them. Let us apply Lemma 4.7 for δ^* to receive $PMV_{\delta^*}(F)$ on $B = B(x, R)$. As a consequence for $D = B(x, \delta^* R)$, $u_k(y) = h(y)$ we obtain

$$\max_D h \leq C \sum_{y \in D} h(y). \tag{4.18}$$

Similarly $PSMV(F)$ yields

$$\min_D h \geq c \sum_{y \in D} h(y). \tag{4.19}$$

The combination of (4.18) and (4.19) gives the elliptic Harnack inequality for the shrinking parameter δ^*. Finally (H) can be shown using the standard chaining argument along a finite chain of balls. The finiteness of the number of balls follows from volume doubling via the bounded covering principle. □

Theorem 4.6. *If (p_0), (VD) hold furthermore there is an $F \in W_0$ for which $PMV(F)$ and $PSMV(F)$ are satisfied, then $E \simeq F$ and (TC) is true.*

Proposition 4.8. *Assume (p_0) and (VD) hold. If $PLE(F)$ holds for $F \in W_0$ hold, then then there is a $c > 0$ such that for all $R > 0$, $x \in \Gamma$*

$$E(x, R) \geq cF(x, R).$$

Proof. It follows from $PLE(F)$ that there are $c, C, 1 > \delta > \delta' > 0$, $1 > \varepsilon > \varepsilon' > 0$ such that for all $x \in \Gamma$, $R > 1$, $A = B(x, 2R)$ and n: $\varepsilon' F(x, R) < n < \varepsilon F(x, R)$, $r = \delta' R$, $y \in B = B(x, r)$

$$\widetilde{P}_n^A(x, y) = P_n^A(x, y) + P_{n+1}^A(x, y) \geq \frac{c\mu(y)}{V(x, R)}.$$

It follows for $F = \varepsilon F(x, R)$, $F' = \varepsilon' F(x, R)$ that

$$\begin{aligned} E(x, 2R) &= \sum_{k=0}^{\infty} \sum_{y \in B(x,2R)} P_k^A(x, y) \geq \sum_{k=0}^{\infty} \sum_{y \in B} \frac{1}{2} \widetilde{P}_k^A(x, y) \\ &\geq \sum_{k=F'}^{F} \sum_{y \in B} \frac{1}{2} \widetilde{P}_k^A(x, y) \geq c \frac{V(x, r)}{V(x, R)} F(x, R) \geq cF(x, R). \end{aligned}$$

□

Proposition 4.9. *If (p_0), (VD) hold and $DUE(F)$ holds for an $F \in W_0$, then there is a $C > 0$ such that for all $R > 0$, $x \in \Gamma$*

$$\rho(x, 2R) v(x, 2R) \leq CF(x, 2R).$$

The first step towards the upper estimate of ρv is to show an upper estimate for λ^{-1}.

Proposition 4.10. *If (p_0), (VD), $DUE(F)$ hold and $F \in W_0$, then there is a $c > 0$ such that for all $R > 0$, $x \in \Gamma$*

$$\lambda(x, R) \geq cF^{-1}(x, R). \tag{4.20}$$

Proof. Assume that $C_1 > 1$, $2n = \lceil F(x, C_1 R) \rceil$, $y, z \in B = B(x, R)$. One can use

$$P_{2n}(y, z) = \sum_w P_n(y, w) P_n(w, z) \leq \sqrt{P_{2n}(y, y) P_{2n}(z, z)}$$

and $DUE(F)$ to get

$$P_{2n}(y, z) \leq C \frac{\mu(z)}{(V(y, f(y, 2n)) V(z, f(z, 2n)))^{1/2}}$$

(for the details see [15]). From (VD) and $F \in W_0$ it follows for $w = y$ or z, $d(x, w) \leq R < C_1 R = f(x, 2n)$

$$\frac{V(x, C_1 R)}{V(w, C_1 R)} \leq C,$$

which results using (p_0) that for all n

$$P_n(y, z) \leq C \frac{\mu(z)}{V(x, f(x, n))}.$$

If ϕ is the left eigenvector (measure) belonging to the smallest eigenvalue λ of $-\Delta^B$ normalized to $(\phi 1) = 1$, then

$$
\begin{aligned}
(1-\lambda)^{2n} &= \phi P_{2n}^B 1 = \sum_{y,z\in B(x,R)} \phi(z) P_{2n}^B(z,y) \le \sum_{y\in B(x,R)} \frac{C\mu(y)}{\min_{z\in B(x,R)} V(z, f(z,2n))} \\
&\le C \max_{z\in B(x,R)} \left(\frac{R}{f(z,2n)}\right)^{\alpha} = C \max_{z\in B(x,R)} \left(\frac{1}{C_1}\frac{f(x,2n)}{f(z,2n)}\right)^{\alpha} \\
&\le C\left(\frac{1}{C_1} C_f\right)^{\alpha} \le \frac{1}{2},
\end{aligned}
$$

if $C_1 = 2C^{1/\alpha} C_f$. Using the inequality and $1-\xi \ge \frac{1}{2}\log\frac{1}{\xi}$ for $\xi \in [\frac{1}{2}, 1]$, where $\xi = 1-\lambda(x,R)$, one has

$$
\lambda(x,R) \ge \frac{\log 2}{4n} \ge cF(x, C_1R)^{-1} > cF(x,R)^{-1}. \qquad \square
$$

Proof of Proposition 4.9. Let us recall from [35] that

$$
\lambda(x,2R)\rho(x,R,2R)V(x,R) \le 1
$$

in general, applying (VD) and (4.20) immediately yields the statement. $\square$

Proposition 4.11. *If (p_0), (VD) hold and $PLE(F)$ for an $F \in W_0$, then there is a $c > 0$ such that for all $R > 0$, $x \in \Gamma$*

$$
\rho(x,R,2R)v(x,R,2R) \ge cF(x,2R).
$$

Proof. The inequality (3.7) states that

$$
\rho(x,R,2R)v(x,R,2R) \ge \min_{z\in\partial B(x,(3/2)R)} E\left(z, \frac{R}{2}\right). \tag{4.21}
$$

From Proposition 4.8 we know that

$$
\min_{z\in\partial B(x,(3/2)R)} E\left(z, \frac{R}{2}\right) \ge c \min_{z\in\partial B(x,(3/2)R)} F\left(z, \frac{R}{2}\right)
$$

and from $F \in W_0$ it follows that

$$
\rho(x,R,2R)v(x,R,2R) \ge \min_{z\in\partial B(x,(3/2)R)} F\left(z, \frac{R}{2}\right) \ge cF(x,2R). \qquad \square
$$

Proof of Theorem 4.6. From Proposition 4.9 we have that $\rho v < CF$ which together with Proposition 4.11 yields that

$$
\rho(x,R,2R)v(x,R,2R) \simeq F(x,2R).
$$

Since $F \in W_0$ we have that $\rho v \in W_0$ and $(aD\rho v)$ as well. From the conditions we have (H) thanks to Theorem 4.5 and by Theorem 3.2 the Einstein relation follows:

$$
E(x,2R) \simeq \rho(x,R,2R)v(x,R,2R) \simeq F(x,2R). \tag{4.22}
$$

Since $F \in W_0$ and $E \simeq F$ it follows that $E \in W_0$ which includes (TC) and of course (wTC) as well and the proof of (4) $\Rightarrow$ (2) of Theorem 4.1 is complete. $\square$

5. The parabolic Harnack inequality

In this section we will prove the following extension of Theorem 1.2.

Theorem 5.1. *If a weighted graph (Γ,μ) satisfies (p_0), then the following statements are equivalent*:

(1) *(VD) hold and there is an $F\in W_1$ such that $g(F)$ is satisfied,*
(2) *(VD), (H) and $(*)$ holds furthermore $E\in V_1$,*
(3) *(VD), (H) and $\rho v\in V_1$,*
(4) *(VD) and $UE(F)$, $PLE(F)$, for an $F\in W_1$ are satisfied,*
(5) *(VD) holds and there is an $F\in W_1$ such that $PMV(F)$ and $PSMV(F)$ are true,*
(6) *there is an $F\in W_1$ such that the two-sided heat kernel estimate holds: there are $C,\beta\geq\beta'>1$, $c>0$ such that for all $x,y\in\Gamma$, $n\geq d(x,y)$*

$$c\frac{\exp[-C(F(x,d)/n)^{1/(\beta'-1)}]}{V(x,f(x,n))}\leq\widetilde{p}_n(x,y)\leq C\frac{\exp[-c(F(x,d)/n)^{1/(\beta-1)}]}{V(x,f(x,n))}\tag{5.1}$$

where $d=d(x,y)$,
(7) *there is an $F\in W_1$ such that $PH(F)$ holds.*

The equivalence of the statements (1)–(5) are based on Theorem 4.1. What is left is to incorporate (6) and (7). In this section we show that the mean value inequalities for $F\in W_1$ are equivalent to the parabolic Harnack inequality and to the two-sided heat kernel estimate (5.1). We will show the following implications:

$$\left.\begin{matrix}PMV\\PSMV\\(VD)\end{matrix}\right\}\implies PH\implies\left\{\begin{matrix}DUE\\DLE\\PSMV\end{matrix}\right\}\implies\left\{\begin{matrix}(VD)\\PMV\\PSMV\end{matrix}\right.,$$

$$\left.\begin{matrix}UE\\PLE\\(VD)\end{matrix}\right\}\iff\left\{\begin{matrix}UE\\LE\end{matrix}\right..$$

Theorem 5.2. *Assume (p_0). Let $F\in W_1$, then the following equivalence holds*:

$$(VD)+PMV(F)+PSMV(F)\iff PH(F).$$

Remark 5.1. *We give direct proof of the statement instead of the ready alternative from* [10]. *In Theorem* 3.10 *of* [10] *the decomposition method shows that for the parabolic Harnack inequality it is enough to show $UE(F)$ and $PLE(F)$ for the Dirichlet heat kernel on $B(x,R)$. Since we know that $PMV\Leftrightarrow UE$ and $PSMV\Leftrightarrow PLE$ the proof is similar but works via the Dirichlet heat kernel estimates. Here we prefer the direct route.*

Proof of Theorem 5.2. The proof consists of several smaller steps.

1. First we show $PH(F)$ for Dirichlet solutions for a particular set of constants. We choose $c_1<\cdots<c_4$ and $\delta^*=\frac{1}{C_F}(c_3-c_2)^{1/\beta'}\delta/2$ as in Lemma 4.7. Denote $\Phi^+=[c_3F,c_4F]\times B(x,\delta^*R)$ and $\Phi^-=[c_1F,c_2F]\times B(x,\delta^*R)$. Using Lemma 4.7 we have for δ^*, $PMV_{\delta^*}(F)$:

$$\max_{\Phi^+}u\leq\frac{C}{\nu(\Phi^-)}\sum_{\Phi^-}u_i(z)\mu(z).\tag{5.2}$$

Let us choose $c_6>c_5>c_4$. The parabolic super mean value inequality $PSMV(F)$ with $\mathcal{D}^+=[c_5F,c_6F]\times B(x,\delta^*R)$, $\mathcal{D}^-=\Phi^-$ states that

$$\min_{\mathcal{D}^+}\widetilde{u}\geq\frac{c}{\nu(\Phi^-)}\sum_{\Phi^-}\widetilde{u}_i(z)\mu(z).\tag{5.3}$$

The combination of (5.2) and (5.3) results that

$$\max_{D^-} u \le C \min_{D^+} \widetilde{u}, \tag{5.4}$$

which is the parabolic Harnack inequality for Dirichlet solutions for the constants $c_3 < c_4 < c_5 < c_6$, δ^*, in other words $D^- = \Phi^+$, $D^+ = \mathcal{D}^+$.

2. Let us use the decomposition for an arbitrary solution $w \ge 0$ on $\mathcal{D} = [0, F(x, R)] \times B(x, R)$. The nonnegative linear decomposition results in a Dirichlet solution $u \ge 0$ on $\mathcal{D}$ for which $u = w$ on $B(x, \delta^* R)$ and $u \le w$ in general (for the details of the decomposition method see [10], proof of Theorem 3.10). Now we use (5.4)

$$\max_{D^-} w = \max_{D^-} u \le C \min_{D^+} u \le C \min_{D^+} w.$$

Which means that we have $PH(F)$ for all solutions and for the given c_i-s and δ^*.

3. It is standard knowledge that if the (classical) parabolic Harnack inequality holds for a set of constants c_i, δ, then it is true for an arbitrary set of constants as well (with another C). This is the case if $F \in W_1$. The key is that $\beta' > 1$ ensures that the time dimension of the space–time cylinder shrinks faster than the space dimension and the usual chaining argument can be applied.
4. The implication $PH(F) \Rightarrow (VD)$ can be seen along the lines of the classical proof (cf. [10]). First from $PH(F)$ the diagonal upper and lower estimates are deduced without change of the proof

$$p_m(x, x) \le \frac{C}{V(x, f(x, m))} \tag{5.5}$$

and

$$\widetilde{p}_n(x, x) \ge \frac{c}{V(x, f(x, n))}. \tag{5.6}$$

The inequality for $n < cm$,

$$p_n(x, x) \le C \widetilde{p}_m(x, x) \tag{5.7}$$

can be obtained from $PH(F)$ with the proper choice of the constants. Now let $n = \lfloor F(x, R) \rfloor$, $m = \lceil F(x, A^p R) \rceil$, $p \ge 1$ and $A \ge 2$ is chosen to satisfy $p > \frac{\beta}{\beta'}$ and $A > (C_F/c_F)^{1/(p\beta' - \beta)}$. As a result from (5.5)–(5.7) one obtains (VD):

$$V(x, 2R) \le V\big(x, A^p R\big) \le C V(x, R).$$

5. The implication $PH(F) \Rightarrow PSMV(F)$ is evident. As in step 4 we deduced $PH(F) \Rightarrow DUE(F)$ and $PMV(F)$ follows from Theorem 4.1. □

Remark 5.2. *The elliptic Harnack inequality is a direct consequence of the F-parabolic one as it is true for the classical case.*

Theorem 5.3. *Assume that (Γ, μ) satisfies (p_0) and (VD). Then for any $F \in W_1$*

$$NDLE(F) \quad \Longrightarrow \quad LE(F).$$

Proof. A modified version of Aronson's chaining argument gives the statement. The proof uses varying radii for the chain of balls. We give the idea of the modification (the other technical details can be seen following [34] or [15]).

Denote δ the constant in $NDLE(F)$ and let $1 > \delta' > 0$ be arbitrary. If $d(x, y) < \delta f(x, n)$ the statement follows from $NDLE$, if $\delta' n \le d(x, y) \le n$ it follows from (p_0).

Assume that $\delta f(x, n) < d(x, y) < \delta' n$. Consider a shortest path π between x and y, denote $d = d(x, y)$,

$$m = \left\lfloor \frac{n}{l(n, R, A)} \right\rfloor - 1, \tag{5.8}$$

$R = f(x,n)$, $S = f(y,n)$, $A = B(x, d+R) \cup B(y, d+S)$. Let $o_1 = x$ and

$$r_1 = \lceil \delta c_0 f(o_1, m) \rceil$$

and choose $o_2 \in \pi\colon\ d(o_1, o_2) = r_1 - 1$ and recursively

$$r_i = \lceil \delta c_0 f(o_i, m) \rceil \tag{5.9}$$

and $o_i \in \pi\colon\ d(o_i, o_{i+1}) = r_i - 1$ and $d(y, o_{i+1}) < d(y, o_i)$. Denote $B_i = B(o_i, r_i)$. The iteration ends for the first j for which $y \in B_j$. From $F \in W_0$ and $z_{i+1} \in B_i$ it follows that

$$c_1 \le \frac{f(z_i, m)}{f(z_{i+1}, m)} \le C_2 \tag{5.10}$$

and from the triangle inequality it is evident that

$$d(z_i, z_{i+1}) \le 2r_i + r_{i+1} \le \left(2 + \frac{1}{c_1}\right)\delta c_0 f(z_i, m). \tag{5.11}$$

Here we specify $c_0 = (2 + 1/c_1)^{-1}$. Let us recall the definition of $l = l(n, d, A)$

$$\frac{n}{l} \ge \max_{z \in A} CE\left(z, \frac{d}{l}\right), \tag{5.12}$$

taking the inverse one obtains:

$$\min_{z \in A} f\left(z, \frac{1}{C}\frac{n}{l}\right) \ge \frac{d}{l}. \tag{5.13}$$

Let us choose C in (5.12) (using $F \in W_1$) such that

$$f\left(o_i, \frac{1}{C}\frac{n}{l}\right) \le \delta c_0 f\left(o_i, \frac{n}{l}\right) = r_i.$$

By the definition of j

$$d > \sum_{i=1}^{j-1} r_i \ge (j-1)\frac{d}{l},$$

consequently, $j - 1 \le l$.

$$(\widetilde{P}_m)^j(x,y) \ge \sum_{z_1 \in B_0} \cdots \sum_{z_{j-1} \in B_{j-2}} \widetilde{P}_m(x, z_1)\widetilde{P}_m(z_1, z_2) \cdots \widetilde{P}_m(z_{j-1}, y).$$

Now we use *NDLE* to obtain

$$\begin{aligned}
(\widetilde{P}_m)^j(x,y) &\ge \sum_{z_1 \in B_0} \cdots \sum_{z_{j-1} \in B_{j-2}} \frac{c\mu(z_1)}{V(x, f(x,m))} \cdots \frac{c\mu(y)}{V(z_{j-1}, f(z_{j-1}, m))} \\
&\ge \min_{z_2 \in B_1} \cdots \min_{z_{j-1} \in B_{j-2}} c^{j-1} \frac{V(o_1, r_1)}{V(x, f(x,m))} \cdots \frac{V(o_{j-1}, r_{j-1})}{V(z_{j-2}, f(z_{j-2}, m))} \frac{\mu(y)}{V(z_{j-1}, f(z_{j-1}, m))} \\
&\ge \min_{z_2 \in B_1} \cdots \min_{z_{j-1} \in B_{j-2}} c^{j-1} \frac{\mu(y)}{V(x, f(x,m))} \frac{V(o_1, r_1)}{V(z_2, f(z_2, m))} \cdots \frac{V(o_{j-2}, r_{j-2})}{V(z_{j-1}, f(z_{j-1}, m))}.
\end{aligned}$$

If we use (5.9), (5.10) and (*VD*) it follows that

$$(\widetilde{P}_m)^j(x,y) \geq \min_{z_2\in B_1}\cdots\min_{z_{j-1}\in B_{j-2}} \frac{c^{j-1}\mu(y)}{V(x,f(x,m))}\frac{V(o_1,r_1)}{V(z_2,1/(\delta c_0c_1)r_1)}\cdots\frac{V(o_{j-2},r_{j-2})}{V(z_{j-1},1/(\delta c_0c_1)r_{j-2})} \tag{5.14}$$

$$\geq \frac{c^{j-1}\mu(y)}{V(x,f(x,m))}\bigl(c'\bigr)^{j-2}$$

$$\geq \frac{c\mu(y)}{V(x,f(x,n))}\exp\bigl[-C(j-1)\bigr]$$

$$\geq \frac{c\mu(y)}{V(x,f(x,n))}\exp[-Cl]. \tag{5.15}$$

From Lemma 13.6 of [15] we know that there is a $c>0$ such that

$$\widetilde{P}_n \geq c^{n-lm}(\widetilde{P}_m)^l$$

if $n\geq lm+l-1$. Let us note that from (5.8) it follows that $n-lm+l\leq 3l$ which results in

$$\widetilde{P}_n(x,y)\geq c^{n-lm}(\widetilde{P}_m)^l(x,y)\geq c'\frac{c^{3l}\mu(y)}{V(x,f(x,n))}\exp(-Cl)$$

$$\geq \frac{c\mu(y)}{V(x,f(x,n))}\exp\left[-C\left(\frac{F(x,d(x,y))}{n}\right)^{1/(\beta'-1)}\right].$$

This finishes the proof of the lower estimate. □

Proof of Theorem 5.1. First of all we have seen that under conditions (*VD*) + (*H*) + (∗) we have that

$$E\in W_0 \quad\text{or}\quad \rho v\in W_0$$

and (*ER*). If in addition $E\in V_1$, then both functions belong to W_1. On the other hand $\rho v\in V_1$ implies (*aD*ρv) which is in the set of equivalent conditions (∗), furthermore $W_0\cap V_1=W_1$ which shows the equivalence of (2) and (3). Based on these observations the equivalence of (1)–(4) and (5) is established by Theorem 4.1. The equivalence (5) and (7) is given in Theorem 5.2.

The implication (4) ⇒ (6) follows from Theorems 4.1 and 5.3. The reverse implication with respect to the upper estimate is also covered by Theorem 4.1 as well. For any $F\in W_1$ $LE(F)$ implies (*VD*). This can be seen exactly as it is proved for $F(x,R)=R^2$. The proof of $UE(F)+LE(F)\Rightarrow PLE(F)$ can be reproduced following the steps of the proof of Lemma 8.3 in [34]. This shows the equivalence of (4) and (6) and proves the whole statement. □

Remark 5.3. *Let us note that we can prove slightly better upper and lower estimates* (*which are*, *in fact*, *equivalent to the ones presented*). *Denote* $d=d(x,y)$. *Following the proof of the upper estimate in* [36] (*see the proof of Theorem* 3.14 *and Remark* 3.4) *one can see that*

$$p_n(x,y)\leq \frac{C\exp[-ck_y(n,(1/2)d)]}{V(x,f(x,n))}+\frac{C\exp[-ck_x(n,(1/2)d)]}{V(y,f(y,n))}. \tag{5.16}$$

The intermediate estimate (5.15) *gives a stronger lower bound*:

$$\widetilde{p}_n(x,y)\geq \frac{c}{V(x,f(x,n))}\exp\bigl[-Cl(x,n,A)\bigr], \tag{5.17}$$

where $A=B(x,d(x,y)+f(x,n))\cup B(y,d(x,y)+f(y,n))$, $n\geq d(x,y)$.

Remark 5.4. *It is not immediate, but is elementary to deduce from* (5.16) *and* (5.17) *a spacial case of Theorem* 5.1 *if*

$$E(x,R)\simeq F(R)$$

or

$$\rho(x,R,2R)v(x,R,2R)\simeq F(R).$$

Such a result is presented in [36]. *The key observation is that under* (p_0), (VD), (H) *the condition* (E) *implies* $\beta'>1$. *The statements* 1–5 *and* 7 *of Theorem* 5.1 *are immediate, the two-sided heat kernel estimate*

$$c\frac{\exp[-Cm(n,d(x,y))]}{V(x,f(n))}\le\widetilde{p}_n(x,y)\le C\frac{\exp[-cm(n,d(x,y))]}{V(x,f(n))} \tag{5.18}$$

needs some preparation (*here* $f(n)$ *is the inverse of* $F(R)$ *again*). *It follows from* (5.16) *and* (5.17) *and from the fact that for any fixed* $C_i>0$, $x\in\Gamma$

$$k_x(C_1n,C_2R)\simeq l_x(C_3n,C_4R)\simeq m(C_5n,C_6R).$$

In the very particular case when $E(x,R)\simeq R^\beta$ *one recovers from* (5.18) *the sub-Gaussian estimate* (1.2) *which is usual for the simplest fractal like graphs*.

List of the main conditions

Shortcut	Equation	Name
(p_0)	(1.5)	controlled weights condition
(VD)	(1.3)	volume doubling property
(TC)	(1.7)	time comparison principle
(ER)	(3.1)	the Einstein relation
(MV)	(2.16)	mean value inequality
$DUE(E)$	(4.3)	diagonal upper estimate
$DLE(F)$	(5.6)	diagonal lower estimate
$g(F)$	(1.16) + (1.17)	bounds on g
(H)	(1.9)	elliptic Harnack inequality
$UE(F)$	(1.18)	upper estimate w.r.t. F
$PLE(F)$	(1.19)	particular lower estimate
$NDLE(F)$	(4.5)	near diagonal lower estimate
$LE(F)$	(1.20)	lower estimate
$PMV(F)$	(1.14)	parabolic mean value inequality
$PSMV(F)$	(1.15)	parabolic super mean value inequality
(E), (ρv)	(2.14) (2.4)	$E(x,R)$ or ρv is uniform in x
$(aD\rho v)$	(2.7)	anti-doubling for ρv
$PH(F)$	(1.14)	parabolic Harnack inequality
$RLEF$	(2.6)	resistance lower estimate

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