Diffusive limits on the Penrose tiling

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Abstract

In this paper random walks on the Penrose tiling and on its local perturbation are investigated. Heat kernel estimates and the invariance principle are shown proving Domokos Szász's conjectures[11].

1 Introduction

The Penrose tiling is an unfailing source of beautiful properties and phenomena to be explored. Domokos Szász [11] formulated the conjecture that the simple nearest neighbor random walk on Penrose graph satisfies the invariance principle. Later he added that local impurities does not destroy that fact. Here we present the proof of the conjectures.

Roughly isometric weighted graphs (c.f. [6]) share diffusion properties. It is enough to refer to the nice paper of Delmotte in which he shows that the two-sided Gaussian heat kernel estimate $(GE_{\alpha,2})$:

$$\frac{c}{n^{\alpha/2}}\exp\left(-C\frac{d\left(x,y\right)^{2}}{n}\right)\leq\widetilde{p}_{n}\left(x,y\right)\leq\frac{C}{n^{\alpha/2}}\exp\left(-c\frac{d\left(x,y\right)^{2}}{n}\right)\qquad(1)$$

is stable under rough isometry for random walks on weighted graphs (here $\alpha\geq1$, $\widetilde{p_n}=p_n+p_{n+1}$). In other words if two graphs are roughly isometric and (1) holds for one then holds for the other as well. We know that (1) holds for the simple symmetric random walk on $\mathbb{Z}^2$ (with $\alpha=2$) consequently holds for the Penrose graph if it is roughly isometric to $\mathbb{Z}^2$. A very short, direct proof will be given of rough isometry between the Penrose graph and $\mathbb{Z}^2$.

Unfortunately the exponents in the upper and lower estimates in (1) contain different constants which reflects that local inhomogeneities affect the diffusion constant for small times ($t<d^2\left(x,y\right)$). The rough isometry

invariance and the Gaussian estimate (1) have been proved along a series of estimates in which some cumulation of constants is unavoidable. Consequently if we are looking for central limit theorem or for the invariance principle we need a different approach.

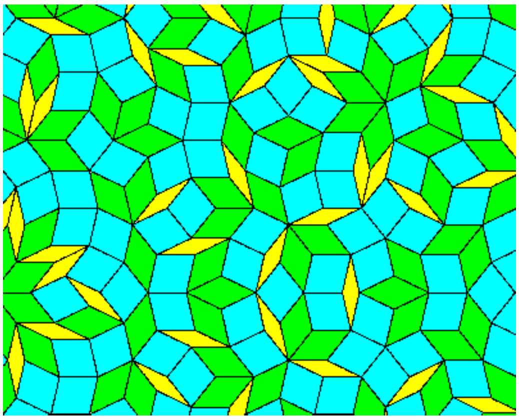
In the field of stochastic processes and statistical physics a powerful method is developed to investigate random walks in random environment, on percolation clusters and interacting particle systems. A key result in this direction is the celebrated paper by Kipnis and Varadhan [8] and its influential extension by De Masi, Ferrari, Goldstein, Wick [4]. The latter one provide us immediate derivation of the central limit theorem and the invariance principle for the random walk on the Penrose net. The proof is inspired by a result by Solomon, [10]. He has shown that the Penrose net is bi-Lipschitz to $\mathbb{Z}^2$. This will be discussed in Section 3 and used in Section 4

Domokos Szász raised the question if invariance principle stabile with respect to local impurities (modification in a bounded region) of the Penrose lattice. The positive answer will be given in Section 5.

In what follows we introduce the basic terminology then each statement is proved in separate section.

2 Preliminaries

We will consider infinite connected graphs with vertex set Γ and edges will be denoted by $x \sim y$. The distance $d(x, y)$ will be the shortest path metric. In particular we will speak about the integer lattice $\mathbb{Z}^m = (\mathbb{Z}^m, d)$ graph where vertexes are element of $\mathbb{Z}^m$ and $x, y \in \mathbb{Z}^m$ form an edge, $x \sim y$, if and only if $|x - y| = 1$. We will speak about the integer lattice net $(\mathbb{Z}^m, |.|)$ if we consider the same vertex and edge set but the metric is the Euclidean one. We do not define the Penrose tiling (c.f. [3] and see Figure 2), we assume that it is well defined and given for us on $\mathbb{R}^2$.



A part of the Penrose tiling ([14])

The Penrose net, $(\Gamma, |.|)$ is a metric space, it is the set of centers of the tiles equipped with the Euclidean distance. Two tiles are neighbors if they are edge adjacent. Two vertexes of the Penrose net are neighbors if they center of neighboring tiles, those vertexes form edges of the net, We will speak about Penrose graph, with the same vertex and edge set but with $d(x, y)$, the shortest path graphs distance. Let $\Gamma = (\Gamma, d_P)$ denote the Penrose graph, and $(\mathbb{Z}^2, d_{\mathbb{Z}})$ integer lattice graph.

We distinguish tilings by fixing a reference vertex and identifying it with the origin of $\mathbb{R}^d$. Let $d(x)$ be the degree of x, the number of neighbors. The random walk on the Penrose net (and graph) is reversible Markov chain with transition probability $P(x, y) = 1/d(x) = 1/4$ for $x \sim y$. It is clear that $d(x) P(x, y) = d(y) P(y, x) = 1$. Denote X_i the actual position of the Markov chain (random walk) which is well-defined for any fixed $X_0 \in \Gamma$.

Definition 1 *A graph is weighted if a symmetric weight function $\mu_{x,y} > 0$ is given on the edges. This weight defines a measure on vertexes and sets:*

$$\mu(x) = \sum_{y \sim x} \mu_{x,y}$$
$$\mu(A) = \sum_{x \in A} \mu(x)$$

Denote $B(x, r) = \{y : d(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$ its volume. In particular the Penrose net (and graph) $\mu_{x,y} \equiv 1$ for edges zero otherwise. The same applies for the integer lattice.

Definition 2 *In general a random walk X_n on Γ with μ is a reversible Markov chain defined by the one step transition probabilities:*

$$P\left(X_{\ n}=y|X_{n-1}=x\right)=P\left(x,y\right)=\frac{\mu_{x,y}}{\mu\left(x\right)}.$$

3 Heat kernel estimate for the Penrose graph

First of all we give the definition the bi-Lipschitz property and rough isometry.

Definition 3 *A metric space (Γ, d) is bi-Lipschitz to (Γ', d') if there is a bijection Φ from Γ to Γ' and a constant $C > 1$ such that for all $x \neq y \in \Gamma$*

$$\frac{1}{C}d\left(x,y\right)\leq d'\left(\Phi\left(x\right),\Phi\left(y\right)\right)\leq Cd\left(x,y\right) \tag{2}$$

Definition 4 *Two weighted graphs Γ with μ and Γ' with μ' are roughly isometric (or quasi isometric) (c.f. [1, Definition 5.9]) if there is a map ϕ from Γ to Γ' such that there are $a, b, c, M > 0$ for which*

$$\frac{1}{a}d\left(x,y\right)-b\leq d'\left(\phi\left(x\right),\phi\left(y\right)\right)\leq ad\left(x,y\right)+b \tag{3}$$

for all $x, y \in \Gamma$,

$$d'\left(\phi\left(\Gamma\right),y'\right)\leq M \tag{4}$$

for all $y' \in \Gamma'$ and

$$\frac{1}{c}\mu\left(x\right)\leq\mu'\left(\phi\left(x\right)\right)\leq c\mu\left(x\right) \tag{5}$$

for all $x \in \Gamma$.

Remark 5 *It is clear that if ϕ from Γ to Γ' is a rough isometry then there is a rough isometry ϕ' from Γ' to Γ as well..*

Theorem 6 *(Solomon [10]) The Penrose net is bi-Lipschitz to the integer lattice net.*

Proposition 7 *The Penrose net is rough isometric to the integer lattice net.*

The statement follows from the bi-Lipschitz property. A very short and direct proof can be given, which we present here.

Proof of 7. Denote m the smaller distance between the opposite boundaries of the thin rhombus and $\varepsilon = \sqrt{2}m/4$. Consider the integer net $\varepsilon\mathbb{Z}^2$ It is clear that if an open square with edge lenght ε contains a center of a rhombus that it is fully contained by the closed rhombus. Let Ψ map the center of the rhombus to the center of the square. It is clear that Ψ is rough isometry from the Penrose net to the integer net. ■

Proposition 8 *The Penrose graph is roughly isometric to the integer lattice graph.*

Proof. Let us consider Ψ, the map introduced above, between Γ and $\varepsilon\mathbb{Z}^2$. Now we consider the graph distances $d_P, d_{\mathbb{Z}}$. It is clear that

$$d_P(x,y) \leq d_{\mathbb{Z}}(\Psi(x), \Psi(y)).$$

The opposite inequality is also easy. Let $2L$ be the maxima number of squares which is needed to cover the largest diagonal of rhombi. It is clear that L is bounded since the diameter of the rhombi is also bounded. Then

$$d_{\mathbb{Z}}(\Psi(x), \Psi(y)) \leq 2L d_P(x,y).$$

It is also clear that the conditions $(4,5)$ are satisfied. ■

Lemma 9 *The Penrose net and graph are rough isometric.*

Proof. Let Φ_1 the rough isometry from the graph Γ to the net Z^2, Φ_2 and the identity map on Z^2 which is bi-Lipschitz between the integer net and graph, finally Φ_3 the rough isometry from the integer graph to the Penrose graph. The existence of Φ_3 follows from Proposition 8 and Remark 5. Then $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$ is rough isometry between the Penrose net and graph. ■

Now we recall Delmotte's result [2] skipping the definition of the parabolic Harnack inequality (PH_2) since we do not need it in the sequel.

Theorem 10 *Let Γ with μ be a weighted graph. Assume that there is a $p_0 > 0$ such that for all edges $P(x,y) \geq p_0$, then the following statements are equivalent.*

1. there are $C, c > 0$, $\alpha \geq 1$ such that for all $x, y \in \Gamma$ and $n > 0$, such that

$$\frac{c}{n^{\alpha/2}} \exp\left(-C\frac{d(x,y)^2}{n}\right) \leq \widetilde{p}_n(x,y) \leq \frac{C}{n^{\alpha/2}} \exp\left(-c\frac{d(x,y)^2}{n}\right) \quad (6)$$

holds,
2.
(i) The volume doubling condition (VD) holds: there is a $C>0$ such that for all $x\in\Gamma$, $r\geq 1$

$$V(x,2r)\leq CV(x,r)$$

and
(ii) the Poincare inequality (PI_2) holds:
there is a $C>0$, such that for all x and, $r>1, f: B(x,r)\rightarrow\mathbb{R}$

$$\sum_{y\in B(x,r)}\left(f(y)-f_B\right)^2\mu(y)\leq cr^2\sum_{y,z\in B(x,r)}\left(f(y)-f(z)\right)^2\mu_{y,z}$$

where $f_B=\frac{1}{V(x,r)}\sum_{y\in B}f(y)\mu(y)$, $f\neq 0$, $B=B(x,r)$
3. The parabolic Harnack inequality (PH_2) holds.

It is well known that the volume doubling property as well as the Poincare inequality are rough isometry invariant. Consequently the properties $(GE_{\alpha,2})$ and (PH_2) are also rough isometry invariant.

Corollary 11 *If Γ with μ and Γ' with μ' are roughly isometric graphs then $(GE_{\alpha,2})$ (and (PH_2)) holds for one if and only if holds for the other.*

Theorem 12 *The Gaussian estimate $(GE_{2,2})$ holds for the random walk on the Penrose graph.*

Proof. Proposition 7 ensures that Penrose graph is rough isometric to $\mathbb{Z}^2$. It is well-known that $(GE_{2,2})$ holds for the random walk on the integer lattice (graph), and then by Corollary 11 $(GE_{2,2})$ holds for the random walk on the Penrose graph as well. ■

4 The invariance principle

In this section we confine ourself to the Penrose net. The Gaussian estimate $(GE_{2,2})$ provides a nice description of the random walk on the Penrose graph but the different constants in the exponents mean that we have only estimate on the variance and it may change from place to place as well as in time. Particularly we do not know if the properly scaled mean square displacement $\frac{1}{n}E\left(d^2(X_0,X_n)\right)$ has a limit. Of course we expect that due to the asymptotic spherical symmetry of the Penrose tiling the diffusion matrix is the identity matrix up to a fixed constant multiplier. In other words the scaled mean square displacement is direction independent. In order to obtain the invariance principle for the Penrose net we need a different method. This is the method of ergodic processes of the environment "seen from the tagged particle" (c.f. [8],[4]). Thanks to the result of De Masi & all [4] it is enough to check that the conditions of Theorem 2.1 in [4] are satisfied and that the covariance matrix is positive definite.

Theorem 13 *The random walk on the Penrose net satisfies the central limit theorem and the invariance principle with non-degenerate covariance matrix.*

Proof. We consider the environment process ω_n seen from the particle. It is more convenient to use (as it is done by Kunz in [7]) the Markov chain $z_n = (\omega_n, X_n)$. Kunz have shown that z_n is ergodic, (see also a more general result by Robinson [9]). It is clear that $X_n = \sum_{i=1}^{n} V(z_{i-1}, z_i)$ where $V(z_{i-1}, z_i) = X_i - X_{i-1}$ is an antisymmetric function, (c.f. [4] (2.3),(2.6) and the remark below it.) It follows from the definition that X_n and z_n as well are reversible. From the ergodicity we have the invariant measure μ and the only properties are left to check are the existence of the conditional drift

$$\varphi = \mathbb{E}_\mu (X_1 - X_0 | X_0) \tag{7}$$

exists and the covariance matrix

$$D = \mathbb{E}_\mu ((X_1 - \varphi)(X_1 - \varphi)^*) \tag{8}$$

is non-degenerate. For any given $X_0 = x$ the conditional drift evidently exists thanks to the bounded distances of neighbors.

Let us recall that the D always exists (see Remark 1. below (2.30) in [4]).

We show that the covariance matrix is positive definite. Let us consider the annulus $B(0, C_2\sqrt{n}) \setminus B(0, C_1\sqrt{n})$ intersected with the cone about a given direction $e \in \mathbb{R}^5$ with angle $\pi/2 > \alpha > 0$. Let H denote the intersection. The constants C_1, C_2 are arbitrary and fixed. Let us recall (Lemma 9) that the Penrose net and graph are roughly isometric

$$\begin{aligned}
& \frac{1}{n}\mathbb{E}(e^* X_n X_n^* E e | X_0 = x_0) \\
= \ & \frac{1}{n}\mathbb{E}\left((e^* X_n)^2 | X_0 = x_0\right) \\
\geq \ & \frac{1}{n}\sum_{x \in H} (ex)^2 P_n(x_0, x) \\
\geq \ & \frac{|H|}{n} c \left(\cos(\alpha)\sqrt{n}\right)^2 \frac{c' \exp\left[-C\frac{\left(aC_2\sqrt{n}+b\right)^2}{n}\right]}{n} \geq c > 0
\end{aligned}$$

independently of x_0 and e, hence the covariance matrix is non-degenerate. By this we have shown the invariance principle holds for the random walk on the Penrose net furthermore the limiting process is a non-degenerate Brownian motion. ■

5 Local perturbation

The Penrose net itself is an aperiodic structure, a quasi-crystal. In crystals impurities may appear and their impact on the diffusion can be dramatic. The simplest example for the anomalous behavior is the simple symmetric random walk on $\mathbb{Z}$ modified at the origin . It is shown in [12] that for the random walk on $\mathbb{Z}^d$ with $d \geq 2$ this not the case, local perturbation of the medium does not destroy the invariance principle.

Definition 14 *Let* Γ *a graph,* $A \subset \Gamma$, *then* $\partial A = \{y \in \Gamma \backslash A : \exists x \in A : y \sim x\}$

Definition 15 $\Gamma\prime$ *is local perturbation of* Γ *if there is a finite set* $A \subset \Gamma$ *such that if we remove* A *from* Γ *and replace it with a finite graph and connect some of its vertices to* ∂A *we receive* Γ'.

Theorem 16 *Assume that* Γ' *is the Penrose net modified in a finite region. Assume that the starting point of the random walk on* Γ' *belongs to the infinite component. Then the invariance principle holds for the random walk on* Γ'. *The covariance matrix is positive definite.*

Proof. Consider the environment process $\omega_n' \in \Omega$ and $Z_n' = (\omega_n', X_n')$ and the function V as above. One can recall from de Bruijn [3] construction or Kuncz paper [7] that in the case of the Penrose net ω-s can be mapped into Ω, union of ten two-dimensional tori of unit equators $\Omega_{i,j}$ $i \neq j \in \{1, 2, ...5\}$. The new environment will be contained in the same state space It is clear that ω_n is again a reversible Markov chain. It is easy to extend the proof of ergodicity given by Kunz to the present case. The main idea is the following. Let us consider the original Penrose tiling and pentagrid. Let $x \in A \cup \partial A$ be a new vertex or its neighbor in $\Gamma \backslash A$, $d(x)$ its degree and y_x the centre of the tile (of the original tiling) which contains it. Then let

$$d\mu'(x) = \frac{d(x)}{4} d\mu(y_x)$$

for those vertices while $d\mu'(x) = d\mu(x)$ for the others. Following Kuncz's argumentation it is clear that μ' is stationary measure with respect to the dynamic defined by the walk on Ω. The new measure μ' differs from the old one only in a zero measure set and the orbit of the dynamic is still countable, have zero measure and dens in Ω again. This means that the original proof of ergodicity applies, particularly since the modification restricted a zero measure set and the process started in the infinite connected part of the graph. The proof uses a non-overlapping packing of Ω with small squares $b_{x_n}, \mu(b_{x_n}) =: \mu(b_0)$ with centres sorted out from possible ω_n-s. We simply drop out those b_{x_n} squares of which the centre corresponds to a vertex in A. This change will causes only $|A \cap \Gamma| \varepsilon \mu(b_0)$ small error, where ε (and $\mu(b_0)$) can be arbitrary small while $|A \cap \Gamma|$ is fixed, bounded. . This yields the invariance principle. Our proof for the non-degeneracy of the covariance matrix works without change. ■

There are many questions left open with respect of the random walk on the Penrose tiling.

Remark 17 *The results presented in this paper carry over easily to other quasicrystals which can be constructed by methods similar to the one produces the Penrose tiling. This applies to generalized Penrose tilings (produced by p-grids), higher dimensional Penrose tilings and stochastic tilings.*

Remark 18

1. *Let us show that local modification does not influence the covariance matrix of the limiting process.*
2. *It seems plausible that with some extra work one can show that the covariance matrix is the identity matrix multiplied with a positive constant. The exact value of the constant ought to be determined as well.*

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