A Quantitative Lusin Theorem for Functions in BV

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November 19, 2013

Abstract

We extend to the BV case a measure theoretic lemma previously proved by DiBenedetto, Gianazza and Vespri ([1]) in $W^{1,1}_{loc}$. It states that if the set where u is positive occupies a sizable portion of a open set E then the set where u is positive clusters about at least one point of E. In this note we follow the proof given in the Appendix of [3] so we are able to use only a $1-$dimensional Poincaré inequality.

1 Introduction

For $\rho > 0$, denote by $K_\rho(y) \subset \mathbb{R}^N$ a cube of edge ρ centered at y. If y is the origin on $\mathbb{R}^N$, we write $K_\rho(0) = K_\rho$. For any measurable set $A \subset \mathbb{R}^N$, by $|A|$ we denote its N-dimensional Lebesgue measure.

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If u is a continuous function in a domain E and $u(x_0) > 0$ for a point $x_0 \in E$ then there is a $r > 0$ such that $u(x) > 0$ in $K_r(x_0) \cap E$. If $u \in C^1$ then we can quantify r in terms of the C^1 norm of u.
The Lusin Theorem says if u is a measurable function in a bounded domain E, than for any $\varepsilon > 0$ there is a continuous function g such that $g = u$ in E except in a small set $V \subset E$ such that $|V| \leq \varepsilon$.
In this note we want to generalize the previous property in the case of mesaurable functions. Very roughly speaking, we prove that if $u \in BV(E)$ and $u(x_0) > 0$ for a point $x_0 \in E$ than for any $\varepsilon > 0$ there is a positive r, that can be quantitatively estimated in terms of ε and the BV norm of u, such that $u(x) > 0$ for any $x \in K_r(x_0) \cap E$ except in a small set $V \subset E$ such that $|V| \leq \varepsilon |K_r(x_0)|$. Obviously we wil state a more precise result in the sequel.
Such kind of result has natural application in regularity theory for solutions to PDE's (see for instance the monography ([2]) for an overview). The first time it was proved in the Appendix of ([3]) in the case of $W^{1,p}(E)$. It was generalized in the case of $W^{1,1}(E)$ in ([1]). Here we combine the proofs of ([3]) and ([1]) in order to generalize this result in BV spaces. Moreover in this note we use a proof based only on 1-dimensional Poincaré inequality. This approach could be useful in the case anisotropic operators where it is likely that will be necessary to develop a new approach tailored on the structure of the operator (a first step in this direction can be found in ([4])).

We prove the following Measure Theoretical Lemma.

Lemma 1.1 Let $u \in BV(K_\rho)$ satisfy

$$\|u\|_{BV(K_\rho)} \leq \gamma \rho^{N-1} \quad \text{and} \quad |[u > 1]| \geq \alpha |K_\rho| \tag{1.1}$$

for some $\gamma > 0$ and $\alpha \in (0, 1)$. Then, for every $\delta \in (0, 1)$ and $0 < \lambda < 1$ there exist $x_o \in K_\rho$ and $\eta = \eta(\alpha, \delta, \gamma, \lambda, N) \in (0, 1)$, such that

$$|[u > \lambda] \cap K_{\eta\rho}(x_o)| > (1 - \delta)|K_{\eta\rho}(x_o)|. \tag{1.2}$$

Roughly speaking the Lemma asserts that if the set where u is bounded away from zero occupies a sizable portion of K_ρ, then there exists at least one point x_o and a neighborhood $K_{\eta\rho}(x_o)$ where u remains large in a large portion of

$K_{\eta\rho}(x_o)$. Thus the set where u is positive clusters about at least one point of K_ρ.
In Section 2, we operate a suitable partition of K_ρ. In Section 3 we prove the result in the case$N = 2$ (an analagous proof works for $N = 1$. We consider more meaningful to prove the result in the less trivial case $N = 2$). In Section 4, by an induction argument, we extend the lemma to any dimension.

2 Proof – A partition of the cube

It suffices to establish the Lemma for u continuous and $\rho = 1$. For $n \in \mathbb{N}$ partition K_1 into n^N cubes, with pairwise disjoint interior and each of edge $1/n$. Divide these cubes into two finite subcollections $\mathbf{Q}^+$ and $\mathbf{Q}^-$ by

$$Q_j \in \mathbf{Q}^+ \qquad \Longleftrightarrow \qquad |[u > 1] \cap Q_j| > \frac{\alpha}{2}|Q_j|$$
$$Q_i \in \mathbf{Q}^- \qquad \Longleftrightarrow \qquad |[u > 1] \cap Q_i| \leq \frac{\alpha}{2}|Q_i|$$

and denote by $\#(\mathbf{Q}^+)$ the number of cubes in $\mathbf{Q}^+$. By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[u > 1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[u > 1] \cap Q_i| > \alpha|K_1| = \alpha n^N|Q|$$

where $|Q|$ is the common measure of the Q_l. From the definitions of the classes $\mathbf{Q}^\pm$,

$$\alpha n^N < \sum_{Q_j \in \mathbf{Q}^+} \frac{|[u > 1] \cap Q_j|}{|Q_j|} + \sum_{Q_i \in \mathbf{Q}^-} \frac{|[u > 1] \cap Q_i|}{|Q_i|} < \#(\mathbf{Q}^+) + \frac{\alpha}{2}(n^N - \#(\mathbf{Q}^+)).$$

Therefore

$$\#(\mathbf{Q}^+) > \frac{\alpha}{2-\alpha} n^N.$$

Consider now a subcollection $\bar{\mathbf{Q}}^+$ of $\mathbf{Q}^+$. A cube Q_j belongs to $\bar{\mathbf{Q}}^+$ if $Q_j \in \mathbf{Q}^+$ and $\|u\|_{BV(Q_j)} \leq \dfrac{2\alpha}{(2-\alpha)n^N}\|u\|_{BV(K_1)}$.
Clearly

$$\#(\bar{\mathbf{Q}}^+) > \frac{\alpha}{2(2-\alpha)} n^N. \tag{2.1}$$

Fix $\delta, \lambda \in (0,1)$. The idea of the proof is that an alternative occurs. Either there is a cube $Q_j \in \bar{\mathbf{Q}}^+$ such that there is a subcube $\tilde{Q} \subset Q_j$ where

$$|[u > \lambda] \cap \tilde{Q}| \geq (1-\delta)|\tilde{Q}| \tag{2.2}$$

or for any cube $Q_j \in \bar{\mathbf{Q}}^+$ there exists a constant $c = c(\alpha, \delta, \gamma, \eta, N)$ such that

$$\|u\|_{BV(Q_j)} \geq c(\alpha, \delta, \gamma, \lambda, N)\frac{1}{n^{N-1}}. \tag{2.3}$$

Hence if (2.2) does not hold for any cube $Q_j \in \bar{\mathbf{Q}}^+$, we can add (2.3) over all such Q_j. Therefore taking into account (2.1), we have

$$\frac{\alpha}{2-\alpha}c(\alpha, \delta, \gamma, N)n \leq \|u\|_{BV(K_1)} \leq \gamma.$$

and for n large enough this fact leads to an evident absurdum.

3 Proof of the Lemma 1.1 when $N = 2$

The proof is quite similar to the one of appendix A.1 of ([3]) to which we refer the reader for more details. For sake of semplicity we will use the same notation of ([3]).

Let $K_{\frac{1}{n}}(x_o, y_o) \in \bar{\mathbf{Q}}^+$. WLOG we may assume $(x_o, y_o) = (0,0)$. Assume that

$$|[u \leq \lambda] \cap K_{\frac{1}{n}}| \geq \delta |K_{\frac{1}{n}}| \qquad \text{and} \qquad \Big|[u > 1] \cap K_{\frac{1}{n}}\Big| > \frac{\alpha}{2}|K_{\frac{1}{n}}| \tag{3.1}$$

$$\|u\|_{BV(K_{\frac{1}{n}})} \leq \frac{2\alpha}{(2-\alpha)n^2}\|u\|_{BV(K_1)}. \tag{3.2}$$

Denote by (x, y) the coordinates of $\mathbb{R}^2$ and, for $x \in (-\frac{1}{2n}, \frac{1}{2n})$ let $\mathfrak{Y}(x)$ the cross section of the set $[u > 1] \cap K_{\frac{1}{n}}$ with lines parallel to y-axis, through the abscissa x, i.e.

$$\mathfrak{Y}(x) \equiv \{y \in (-\frac{1}{2n}, \frac{1}{2n}) \text{ such that } u(x, y) > 1\}.$$

Therefore

$$|[u > 1] \cap K_{\frac{1}{n}}| \equiv \int_{-\frac{1}{2n}}^{\frac{1}{2}n} |\mathfrak{Y}(x)| dx.$$

Since, by (3.1), $|[u > 1] \cap K_{\frac{1}{n}}| > \frac{\alpha}{2} |K_{\frac{1}{n}}|$,
there exists some $\tilde{x} \in (-\frac{1}{2n}, \frac{1}{2n})$ such that

$$|\mathfrak{Y}(\tilde{x})| \geq \frac{\alpha}{4n}. \tag{3.3}$$

Define

$$A_{\tilde{x}} \equiv \{y \in \mathfrak{Y}(\tilde{x}) \text{ such that } \exists x \in (-\frac{1}{2n}, \frac{1}{2n}) \text{ such that } u(x, y) \leq \frac{(1+\lambda)}{2}\}.$$

Note that for any $y \in A_{\tilde{x}}$ the variation along the x direction is at least $\frac{(1-\lambda)}{2}$. If $|A_{\tilde{x}}| \geq \frac{\alpha}{8n}$, we have that the BV norm of u in $K_{\frac{1}{n}}$ is at least $\frac{\alpha(1-\lambda)}{16n}$ and therefore (2.3) holds.
If $|A_{\tilde{x}}| \leq \frac{\alpha}{8n}$, we have that there exists at least a $\tilde{y} \in \mathfrak{Y}(\tilde{x})$ such that u(x,$\tilde{y}$) $\geq \frac{(1+\lambda)}{2}$ for any $x \in (-\frac{1}{2n}, \frac{1}{2n})$.
Define

$$A_{\tilde{y}} \equiv \{x \in (-\frac{1}{2n}, \frac{1}{2n}) \text{ such that } \exists y \in (-\frac{1}{2n}, \frac{1}{2n}) \text{ such that } u(x, y) \leq \lambda\}.$$

Note that for any $x \in A_{\tilde{y}}$ the variation along the y direction is at least $\frac{(1-\lambda)}{2}$.
If $|A_{\tilde{y}}| \geq \frac{\delta}{n}$ we have that the BV norm of u in $K_{\frac{1}{n}}$ is at least $\frac{\delta(1-\lambda)}{2n}$ and therefore (2.3) holds.
If $|A_{\tilde{y}}| \leq \frac{\delta}{n}$ we have that $|[u > \lambda] \cap K_{\frac{1}{n}}| \geq (1-\delta)|K_{\frac{1}{n}}|$ and therefore (2.2) holds.
Summarasing either (2.2) or (2.3) hold. Therefore the alternative occurs and the case $N = 2$ is proved.

4 Proof of the Lemma 1.1 when $N > 2$

Assume that Lemma 1.1 is proved in the case $N = m$ and let us prove it when $N = m + 1$.
Let z a point of $\mathbb{R}^{m+1}$. To make to notation easier, write $z = (x, y)$ where $x \in \mathbb{R}$ and $y \in \mathbb{R}^m$.
Let $K_{\frac{1}{n}}(z) \in \bar{\mathbf{Q}}^+$. WLOG we may assume $z = (0, 0)$. Assume that

$$|[u \le \lambda] \cap K_{\frac{1}{n}}| \ge \delta |K_{\frac{1}{n}}| \qquad \text{and} \qquad \left|[u > 1] \cap K_{\frac{1}{n}}\right| > \frac{\alpha}{2}|K_{\frac{1}{n}}| \tag{4.1}$$

$$\|u\|_{BV(K_{\frac{1}{n}})} \le \frac{2\alpha}{(2-\alpha)n^{m+1}}\|u\|_{BV(K_1)}. \tag{4.2}$$

For any $x \in (-\frac{1}{2n}, \frac{1}{2n})$ consider the m -dimensional cube centered in $(x, 0)$, orthogonal to the $x-$axis and with edge $\frac{1}{n}$ and denote this cube $\bar{K}_{\frac{1}{n}}(x)$. Define $\bar{A}$ as the set of the $x \in (-\frac{1}{2n}, \frac{1}{2n})$ such that

$$\left|[u > 1] \cap \bar{K}_{\frac{1}{n}}(x)\right| > \frac{\alpha}{4}|\bar{K}_{\frac{1}{n}}(x)|$$

and

$$\|u\|_{BV(\bar{K}_{\frac{1}{n}}(x))} \le \frac{16}{(2-\alpha)n^m}\|u\|_{BV(K_1)}.$$

It is possible to prove that

$$|\bar{A}| \ge \frac{\alpha}{8n}.$$

Let $\bar{x} \in \bar{A}$ and apply Lemma 1.1 to $\bar{K}_{\frac{1}{n}}(\bar{x})$ (we can do so because $\bar{K}_{\frac{1}{n}}(\bar{x})$ is a m-dimensional set).
So we get the existence of a constant $\eta_0 > 0$ and a point $y_o \in \bar{K}_{\frac{1}{n}}(\bar{x})$ such that if we define the set

$$A \equiv \{(\bar{x}, y) \in \bar{K}_{\frac{\eta_0}{n}}(\bar{x}, y_0) \text{ such that } u(\bar{x}, y) \ge \frac{(1+\lambda)}{2}\}$$

where $\bar{K}_{\frac{\eta_0}{n}}(\bar{x}, y_0)$ denotes the $m-$dimensional cube of edge $\frac{\eta_0}{n}$, centered in $(\bar{x}, y_0)$ and orthogonal to the $x-$axis, we have

$$|A| \ge (1 - \frac{\delta}{2})(\frac{\eta_0}{n})^m. \tag{4.3}$$

Define

$$B \equiv \{y \in A \text{ such that } \exists x \in (-\frac{1}{2n}, \frac{1}{2n}) \text{ such that } u(x,y) \leq \lambda\}.$$

Note that for any $y \in B$ the variation along the x direction is at least $\frac{(1-\lambda)}{2}$.
If $|B| \geq \frac{\delta}{2}(\frac{\eta_0}{n})^m$, we have that the BV norm of u in $K_{\frac{1}{n}}$ is at least $\frac{\delta(1-\lambda)}{4}(\frac{\eta_0}{n})^m$ and therefore (2.3) holds.
If $|B| \geq \frac{\delta}{2}(\frac{\eta_0}{n})^m$, taking in account (4.3) we have that in the cylinder $(-\frac{1}{2n}, \frac{1}{2n}) \times \bar{K}_{\frac{\eta_0}{n}}(0, y_0)$ the measure of the set where $u(x,y) \geq \lambda$ is greater than $(1-\delta)\frac{\eta_o^m}{n^{m+1}}$. Therefore (2.2) holds in a suitable subcube of $K_{\frac{1}{n}}$.
Summarasing either (2.2) or (2.3) hold. Therefore the alternative occurs and the case $N > 2$ is proved.

Acknowledgments: This research was supported by the Italian-Hungarian executive project HU11MO10 " NonLinear Diffusion Processes and Mathematical Modelling in Finance".

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