Harnack inequalities and sub-Gaussian estimates for random walks

Alexander Grigor'yan,
Department of Mathematics
London SW7 2BZ
United Kingdom 662
a.grigoryan@ic.ac.uk

Andras Telcs*
IMC Graduate School of Business
Zrinyi u. 14, Budapest
H-1051, Hungary
h197tel@ella.hu

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Abstract

We show that a β-parabolic Harnack inequality for random walks on graphs is equivalent, on one hand, to so called β-Gaussian estimates for the transition probability and, on the other hand, to the conjunction of the elliptic Harnack inequality, the doubling volume property, and the fact that the mean exit time in any ball of radius R is of the order R^β. The latter condition can be replaced by a certain estimate of a resistance of annuli.

Contents

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1 Introduction

In 1986, P.Li and S.-T.Yau [27] proved the following remarkable Gaussian estimate for the heat kernel $p_t(x,y)$ on any complete Riemannian manifold M with non-negative Ricci curvature:

$$p_t(x,y) \simeq \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{d^2(x,y)}{ct}\right). \tag{1.1}$$

Here $d(x,y)$ is the geodesic distance between points $x,y \in M$, $V(x,r)$ is the Riemannian volume of a geodesic ball $B(x,r)$ of radius r centered at x; the sign $\simeq$ means that the ratio of the two sides of (1.1) is bounded from above and below by two positive constants, for all $x,y \in M$ and $t > 0$ (the value of the constant c may be different for the upper and the lower bounds). This estimate reflects the fact that the Brownian motion X_t on the manifold M travels at distance $\approx \sqrt{t}$ over time t.

Many efforts were made to understand what geometric properties of the manifold are responsible for (1.1). The first author [18] and L.Saloff-Coste [31] obtained (1.1) under the assumption that M satisfies *the volume doubling property*: for all $x \in M$ and $R > 0$

$$V(x,2R) \le CV(x,R), \tag{VD}$$

and a certain *Poincaré inequality*. Moreover, L.Saloff-Coste proved that (VD) and the Poincaré inequality are equivalent to (1.1).

For the purposes of the present paper it is important to mention that (1.1) is equivalent to a *parabolic Harnack inequality* (see for example [15]) which says the following: for any non-negative function $u(x,t)$ solving the heat equation $\frac{\partial u}{\partial t} = \Delta u$ in a cylinder $B(z,2R) \times (0,4R^2)$,

$$\sup_{B(z,R)\times(R^2,2R^2)} u(x,t) \le C \inf_{B(z,R)\times(3R^2,4R^2)} u(x,t), \tag{PH_2}$$

where the constant C does not depend on z, R. Clearly, the same space/time scaling *time* $\approx$ *distance*^{2} appears here as well.

The parabolic Harnack inequality implies the *elliptic Harnack inequality*: for any non-negative harmonic function $u(x)$ in a ball $B(z,2R)$,

$$\sup_{B(z,R)} u \le C \inf_{B(z,R)} u. \tag{H}$$

Historically, the first significant results on elliptic and parabolic Harnack inequalities are due to J.Moser [28], [29], who proved them for solutions of uniformly elliptic and parabolic equations in the divergence form in $\mathbb{R}^n$. In the geometric terms the results of Moser mean that (PH_2) and (H) hold on any Riemannian manifold quasi-isometric to $\mathbb{R}^n$.

Moser's method of proving Harnack inequalities as well as the competing methods ([15], [18], [25], [31] and many others) yield (PH_2) and (H) under the same set of assumptions about M. For quite a long time it was not clear if (H) was actually weaker than (PH_2). A solution to this problem arose from a different field − analysis on fractals.

By fractals we mean sets like a Sierpinski carpet, which can be constructed by using certain self-similar procedures. Normally fractals can be equipped with a distance function d, a (Hausdorff) measure μ, and an energy functional $\mathcal{E}$. Denote by M such a metric-measure-energy space. Using methods of abstract potential theory, one defines a Hunt process X_t on M (see [16]), which in many interesting cases happens to be diffusion with a continuous heat kernel $p_t(x,y)$ (see [1]).

M.Barlow and A.Perkins [7], M.Barlow and R.Bass [2], [3], [4] showed that, for a large variety of fractal sets, the following heat kernel estimate takes place:

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{ct}\right)^{\frac{1}{\beta-1}}\right), \tag{1.2}$$

for a certain natural range of x, y, t. Here α, β are parameters related by $2 \le \beta \le \alpha + 1$. Barlow and Bass [4] also showed that on such fractals a *β-parabolic Harnack inequality* (PH_β) holds. This inequality is a generalization of (PH_2) which one obtains by replacing everywhere R^2 by R^β. The existence of spaces satisfying (PH_β) with $\beta > 2$ proves that (H) is actually weaker than (PH_2) because $(PH_\beta) \Longrightarrow (H)$ for *any* β whereas (PH_β) are not equivalent for different β. However, a new interesting question arises:

What is "the difference" between (PH_β) and (H)?

This question is very much related to obtaining criteria for a *β-Gaussian estimate* of the heat kernel:

$$p_t(x, y) \simeq \frac{1}{V(x, t^{1/\beta})} \exp\left(-\left(\frac{d^\beta(x, y)}{ct}\right)^{\frac{1}{\beta-1}}\right). \tag{1.3}$$

In the case of fractals satisfying (1.2) one has $V(x, r) \simeq r^\alpha$. Hence, (1.3) unifies (1.1) and (1.2). Observe that the parameter β governs the space/time scaling by $time = distance^\beta$. In the case $\beta > 2$, the propagation of X_t is slower than in the Gaussian case $\beta = 2$.

The present paper answers the above questions in the setting of random walks on graphs (it is well understood that a large scale behavior of random walks exhibits the same phenomena as that of diffusions on manifolds or fractals). Note first that the volume doubling condition (VD) and the elliptic Harnack inequality (H) are necessary for (PH_β). These two conditions ensure a certain homogeneity of the space M but neither of them contains a parameter β. To recover β one needs a third condition. Let $E(x, R)$ be the *mean exit time* of the process X_t from the ball $B(x, R)$ provided $X_0 = x$. For example in $\mathbb{R}^n$ we have $E(x, R) \simeq R^2$. Consider the hypothesis

$$E(x, R) \simeq R^\beta. \tag{E_β}$$

Then our main result says that (in the setting of random walks)

$$(1.3) \Longleftrightarrow (PH_\beta) \Longleftrightarrow (VD) + (H) + (E_\beta).$$

Alternatively, the condition (E_β) can be replaced by a certain estimate of the *resistance* of annuli (see Sections 2, 3 below). In many interesting cases, (VD) and (E_β) (or the resistance condition) can be effectively verified. However, verifying the elliptic Harnack inequality (H) may be very difficult, and it is still an open problem to find optimal criteria for (H).

Technically speaking, the main novelty of this paper is in obtaining the *on-diagonal upper estimate*

$$p_t(x, x) \le \frac{C}{V(x, t^{1/\beta})} \tag{DUE_β}$$

from $(VD) + (H) + (E_\beta)$. For an off-diagonal upper bound as well as for a lower bound, we use previously known arguments. The major difficulty in proving (DUE_β) is that the right hand side depends on x. A celebrated method of Nash [30] that was successfully applied for proving x-independent estimates like

$$p_t(x, x) \le f(t) \tag{1.4}$$

(see [35], [8], [19], [10]), does not work in our case. The estimate of Li and Yau (1.1) can be proved using a certain parabolic mean value inequality (see [19], [26]). However, all known proofs of the latter are too much linked to the classical Gaussian time/space scaling and do not work in the case $\beta > 2$. This difficulty was overcome by the second author [33], [34] by using an on-diagonal upper bound of the resolvent but this worked only in a strongly recurrent case (note that $\mathbb{R}^n$ is strongly recurrent only for $n = 1$).

To prove (DUE_β) in a full generality, we develop in this paper a new method (partly inspired by [1] and [33]) based on a certain estimate (5.22) for *λ-polyharmonic functions*. This estimate can be considered as an extension of the maximum principle for harmonic functions, and it is proved by a virtue of a higher order Feynman-Kac formula (5.15) (see Section 5).

In Section 2 we describe the setting of random walks on graphs and give necessary definitions. In Section 3 we state the main Theorem 3.1 and discuss some consequences and examples. Sections 4 – 6 contain the proofs. We use the letters c, C to denote positive constants whose values are unimportant and may change at each occurrence.

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2 Preliminaries

The results cited here can be found in many places (cf. [9], [11], [22]).

Measure and distance. Let Γ be an infinite connected locally finite graph endowed with a *weight* μ_{xy}. The latter is a symmetric non-negative function on $\Gamma\times\Gamma$ such that $\mu_{xy}>0$ if and only if x and y are connected by an edge (in which case we write $x\sim y$). The weight μ_{xy} on edges induces a weight $\mu(x)$ on vertices and hence a measure μ on subsets $A\subset\Gamma$ defined by

$$\mu(x):=\sum_{y:y\sim x}\mu_{xy}\quad\text{and}\quad\mu(A):=\sum_{x\in A}\mu(x).$$

Let $d(x,y)$ be the graph distance between the points $x,y\in\Gamma$, that is the minimal number of edges in any edge path connecting x and y. Denote metric balls and their measures as follows:

$$B(x,R):=\{y\in\Gamma:d(x,y)<R\}\quad\text{and}\quad V(x,R):=\mu(B(x,R))\,.$$

A weighted graph (Γ,μ) satisfies the *volume doubling property* if

$$V(x,2R)\leq CV(x,R)\quad\forall x\in\Gamma,\ \forall R>0 \tag{VD}$$

for some constant C. It is known that (VD) self-improves to the following inequality

$$\frac{V(x,R)}{V(y,r)}\leq C\left(\frac{d(x,y)+R}{r}\right)^{\alpha}, \tag{2.1}$$

for all pairs of balls $B(x,R)$ and $B(y,r)$ such that $r\leq R$ and for some α,$C>0$.

Random walk. The main object of our study is a random walk X_n on Γ, which is a reversible Markov chain with respect to μ, defined by the following one-step transition probability

$$P(x,y)=\frac{\mu_{xy}}{\mu(x)}.$$

We also consider $P(x,y)$ as a Markov operator which acts on functions on Γ by $Pf(x)=\sum_y P(x,y)f(y)$. For any non-negative integer n, the n-step transition probability P_n is defined by $P_n(x,y)=\mathbb{P}_x(X_n=y)$. Alternatively, P_n is the n-th convolution power of the operator P. Define the *heat kernel* of (Γ,μ) by

$$p_n(x,y)=\frac{P_n(x,y)}{\mu(y)}. \tag{2.2}$$

It is easy to see that $p_n(x,y)=p_n(y,x)$, and for any $y\in\Gamma$, the function $u_n(x)=p_n(x,y)$ satisfies the *heat equation*

$$u_{n+1}(x)-u_n(x)=\Delta u_n. \tag{2.3}$$

Here Δ is the Laplace operator of the graph (Γ,μ) defined by $\Delta=P-\mathrm{I}$, that is

$$\Delta u(x)=\sum_y P(x,y)u(y)-u(x)=\frac{1}{\mu(x)}\sum_y\left(u(y)-u(x)\right)\mu_{xy}.$$

For any parameter $\beta>1$, consider the following estimates of the heat kernel, which in general may be true or not: for all $x,y\in\Gamma$

$$p_n(x,y)\leq\frac{C}{V(x,n^{1/\beta})}\exp\left[-\left(\frac{d(x,y)^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right],\ \forall n\geq 1, \tag{UE_β}$$

$$(p_n+p_{n+1})(x,y)\geq\frac{c}{V(x,n^{1/\beta})}\exp\left[-\left(\frac{d(x,y)^\beta}{cn}\right)^{\frac{1}{\beta-1}}\right],\quad \forall n\geq d(x,y)\vee 1. \tag{LE_β}$$

The conjunction $(UE_\beta)+(LE_\beta)$ is a discrete analogue of the estimate (1.3) from Introduction. The obvious distinctions between (UE_β) and (LE_β) – the restriction $n\geq d(x,y)$ and the term p_n+p_{n+1} instead of p_n – reflect the discreteness of the time (see [22, Section 14]). Although a priori β is any number >1, in fact $(UE_\beta)+(LE_\beta)$ imply $\beta\geq 2$.

In some parts of this paper, we assume that the following condition is satisfied: for some positive p_0

$$P(x,y)\geq p_0\quad\text{for all } x\sim y. \tag{p_0}$$

In particular, (p_0) implies that each point $x\in\Gamma$ has a uniformly bounded number of edges.

Green function and the killed random walk. For any finite non-empty subset $A\subset\Gamma$, denote by $c_0(A)$ the set of all functions on A extended by 0 outside A. Define operator Δ^A on $c_0(A)$ as follows: for any $f\in c_0(A)$

$$\Delta^A f(x)=\begin{cases}\Delta f(x), & x\in A,\\ 0, & x\notin A.\end{cases}$$

The operator $-\Delta^A$ is positive definite and has the inverse $G^A=\left(-\Delta^A\right)^{-1}$ which is called the *Green function.* Respectively, the *Green kernel* is defined by

$$g^A(x,y)=\frac{G^A(x,y)}{\mu(y)}.$$

Note that $g^A(x,y)$ is non-negative and vanishes if one of the points x,y is outside A. Also, $g^A(x,y)=g^A(y,x)$, and $g^A(\cdot,y)$ satisfies the equation

$$\Delta g^A=-\delta_y$$

where $\delta_y(x)=0$ if $x\neq y$ and $\delta_y(y)=1/\mu(y)$.

Let P^A be restriction of the Markov operator P to the set A. The iterates of P^A are denoted by P_n^A and define the random walk X_n^A with the killing condition outside A. Then we have

$$G^A(x,y)=\sum_{n=0}^\infty P_n^A(x,y)\quad\text{and}\quad g^A(x,y)=\sum_{n=0}^\infty p_n^A(x,y),$$

where p_n^A is the density of P_n^A.

Mean exit time. The exit time from a set $A\subset\Gamma$ is defined as

$$T_A:=\min\{n\geq 0: X_n\notin A\}.$$

Its expectation $\mathbb{E}_x(T_A)$ is called the *mean exit time* and will be denoted also by $E_x(A)$. If $A = B(x,R)$ then we write

$$T_{x,R} := T_{B(x,R)} \quad \text{and} \quad E(x,R) := \mathbb{E}_x\left(T_{B(x,R)}\right).$$

The mean exit time is related to the Green function by the identity

$$E_x(A) = \sum_y G^A(x,y) = \sum_y g^A(x,y)\mu(y). \tag{2.4}$$

Denote

$$\overline{E}(A) := \sup_z E_z(A) \quad \text{and} \quad \overline{E}(x,R) := \overline{E}\left(B(x,R)\right) = \sup_{z\in B(x,R)} \mathbb{E}_z\left(T_{B(x,R)}\right). \tag{2.5}$$

The following two conditions will be frequently used:

$$E(x,R) \simeq R^\beta, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1, \tag{E_β}$$

and

$$\overline{E}(x,R) \leq CE(x,R), \quad \text{for all } x \in \Gamma \text{ and } R > 0. \tag{$\overline{E}$}$$

Resistance. For any function f on Γ define its energy by

$$\mathcal{E}(f) = \frac{1}{2}\sum_{x\sim y}\left(f(x) - f(y)\right)^2 \mu_{xy}.$$

For any two sets $A \subset B \subset \Gamma$ the *resistance* $\rho(A,B)$ is defined by

$$\rho^{-1}(A,B) = \inf\left\{\mathcal{E}(f) : f|_A = 1 \text{ and } f|_{B^c} = 0\right\}, \tag{2.6}$$

where $B^c := \Gamma \setminus B$. The following condition is an analog of (E_β) for resistance:

$$\rho\left(B(x,R), B(x,MR)\right) \simeq \frac{R^\beta}{V(x,R)}, \quad \text{for all } x \in \Gamma \text{ and } R \geq 1, \tag{ρ_β}$$

for some (large) number M whose value is unimportant.

Harnack inequalities. A function u is said to be *harmonic* in a set $A \subset \Gamma$ if u is defined in $\overline{A}$ (that consists of all points in A and all their neighbors) and if $\Delta u(x) = 0$ for any $x \in A$. We says that (Γ,μ) satisfies an *elliptic Harnack inequality* if, for all $x \in \Gamma$, $R > 0$, and for any non-negative harmonic function u in $B(x,2R)$, the following inequality holds

$$\max_{B(x,R)} u \leq C \min_{B(x,R)} u. \tag{H}$$

Given $\beta > 1$, we say that (Γ,μ) satisfies *β-parabolic Harnack inequality* if, for all $x \in \Gamma$, $R \geq 1$ and for any non-negative function $u_n(y)$ defined for $n \in [0,4N]$, $y \in \overline{B(x,2R)}$ and satisfying the heat equation (2.3) in $[0,4N) \times B(x,2R)$, the following inequality holds

$$\max_{\substack{n\in[N,2N)\\ y\in B(x,R)}} u_n(y) \leq C \min_{\substack{n\in[3N,4N)\\ y\in B(x,R)}} \left(u_n(y) + u_{n+1}(y)\right), \tag{PH_β}$$

where N is a positive integer such that $N \simeq R^\beta$ and $N \geq 2R$.

Since any harmonic function satisfies also the heat equation, $(PH_\beta) \Longrightarrow (H)$ for any β.

Remark 2.1 If the graph (Γ,μ) satisfies the condition (p_0) then the Harnack inequality (H) automatically holds for all balls with *bounded* radii R and for all non-negative functions u in $\overline{B(x,R)}$ which are *superharmonic* in $B(x,R)$ (see [22, Proposition 3.2]). The main point of (H) (and (PH_β)) is that it holds for arbitrarily large R with *the same* constant C.

3 Main result

Theorem 3.1 *If graph* (Γ, μ) *satisfies* (p_0) *then, for any* $\beta \geq 2$, *the following conditions are equivalent:*

(i) $(UE_\beta) + (LE_\beta)$

(ii) (PH_β)

(iii) $(VD) + (H) + (\rho_\beta)$

(iv) $(VD) + (H) + (E_\beta)$

When it is not ambiguous we will drop the subscript β from the notation.

Some of the implications in Theorem 3.1 were already known and are included here for completeness as for instance, $(i) \Longleftrightarrow (ii)$ – see [15], [12] for the case $\beta = 2$, and [23] for any β (see also [4]). The implication $(ii) \Longrightarrow (iii)$ was proved in [33, Section 7].

The implication $(iii) \Longrightarrow (iv)$ is proved in the present paper in Section 4.3. In the view of (ρ) and (E), this can be regarded as a proof of the following relation

$$E(x, R) \simeq \rho\left(B(x, R), B(x, MR)\right) V(x, R)$$

which we call *Einstein's relation*[1].

The major part of this paper is devoted to the proof of (UE) in the implication $(iv) \Longrightarrow (i)$. The proof consists of many steps and goes via an upper estimate (G) for the resolvent and a diagonal upper bound (DUE) for the heat kernel as is shown on the diagram:

$$\left.\begin{matrix}(VD)\\(H)\\(E)\end{matrix}\right\} \overset{Thm.5.7}{\Longrightarrow} \left.\begin{matrix}(VD)\\(G)\end{matrix}\right\} \overset{Thm.6.1}{\Longrightarrow} \left.\begin{matrix}(VD)\\(DUE)\\(E)\end{matrix}\right\} \overset{Thm.6.2}{\Longrightarrow} (UE)$$

(see also Section 6.3). The proof of (LE) from $(VD) + (H) + (E)$ is practically contained in [22, Section 13] and is briefly outlined in Section 6.3.

Note that the equivalence $(PH_\beta) \Longleftrightarrow (VD) + (H) + (E_\beta)$ is new even for $\beta = 2$. For the case $\beta = 2$, we have a further result. Denote by $\lambda(x, R)$ the smallest eigenvalue of $-\Delta^{B(x,R)}$ and let (λ) stand for the following conditions: for all $x \in \Gamma$ and $R \geq 1$,

$$\lambda(x, R) \geq cR^{-2}.$$

Corollary 3.2 *If* (Γ, μ) *satisfies* (p_0) *then* $(PH_2) \Longleftrightarrow (VD) + (H) + (\lambda)$.

Indeed, the direction " $\Longrightarrow$ " follows from Theorem 3.1(iv) and the estimate $\lambda_1(x, R) \geq \overline{E}(x, R)^{-1}$ (see [22, Propositions 6.1,6.2]), whereas the direction " $\Longleftarrow$ " follows from the estimates $\lambda(x, 2R)^{-1} \geq \rho\left(B(x, R), B(x, 2R)\right) V(x, R)$ and $\rho\left(B(x, R), B(x, 2R)\right) V(x, 2R) \geq R^2$ (see [33]), and Theorem 3.1(iii).

The proof of Theorem 3.1 contains the following new implication (see Remark 6.1):

$$(VD) + (MV) + (E_\beta) \Longrightarrow (UE_\beta), \tag{3.1}$$

[1] In his celebrated work [14], A.Einstein proved the following formula for a diffusion constant: $D^{-1} = \sigma\mu$ where σ is the resistance density and μ is the mass density of the media in which the diffusion takes place. Much later in the fractal literature (see for example, [17]) the renormalisation technique led to the use of the scaling terms $D^{-1} \approx L^\beta$, $\mu \approx L^\alpha$, $\sigma \approx L^\gamma$ and to the observation that on large families of fractals $\beta = \alpha + \gamma$, conjectured in [32] to be a very general law.

where (MV) is an elliptic mean value inequality defined in Section 5.2. It is interesting to recall that (UE_2) is equivalent to (VD)+a *parabolic* version of (MV) (see [11], [26]). Theorem 3.1 recovers also the results obtained in [22] and [33].

In conclusion of this section, we give some examples highlighting the relations between the hypotheses in questions.

Example 3.1 The hypotheses (VD), (H), (E_β) used in the major part of this paper, are independent. Indeed, $(VD)+(H)$ does not imply (E_β) since there are graphs with $(VD)+(H)+(E_\beta)$ with different β (see [5], [24]). In an example of Delmotte [13], $(H)+(E_\beta)$ holds but not (VD). Finally, let us sketch an example where $(VD)+(E_\beta)$ holds but (H) does not. Indeed, take two copies Γ_1 and Γ_2 of the same graph that satisfies $(VD)+(H)+(E_\beta)$, and glue Γ_1 and Γ_2 together by a single edge obtaining a new graph Γ. Then it is not difficult to prove that $(VD)+(E_\beta)$ holds on Γ but (H) does not, provided the random walk on Γ_i is transient (see [21] for a manifold analogue of this construction). In addition, it is possible to prove that (UE_β) holds on Γ; hence (LE_β) must fail.

Example 3.2 Here we describe a graph satisfying $(VD)+(H)+(E_\beta)$ and such that $V(x,R)$ substantially depends on x. Let Γ be the Vicsek tree (embedded in $\mathbb{R}^2$) that is the union of the increasing sequence of blocks $\{\Gamma_k\}_{k=1}^\infty$ – see Fig. 1. Here $\Gamma_0=\{o\}$, and Γ_{k+1} consists of Γ_k and its four copies translated and glued in an obvious way.

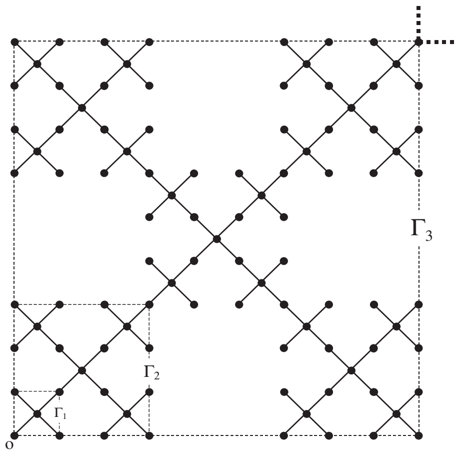

Figure 1 Blocks $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$ in the Vicsek tree

Fix $a \geq 1$ and define weight μ_{xy} for any edge $\overline{xy}$ by $\mu_{xy}=a^k$ where k is the minimal index such that Γ_k contains x or y. Since $d(x,o)\simeq 3^k$ for any $x\in\Gamma_k\setminus\Gamma_{k-1}$, this implies for all $x\neq o$,

$$\mu(x)\simeq d(x,o)^\delta \tag{3.2}$$

where $\delta=\log_3 a$.

Let x_k be the symmetry center of Γ_k and set $R_k=3^{k-1}+\frac{1}{2}$; then $\Gamma_k=B(x_k,R_k)$. Clearly,

$$|B(x_k,R_k)|=|\Gamma_k|\simeq 5^k\simeq R_k^\alpha$$

where $|\cdot|$ is the cardinality of a set and $\alpha=\log_3 5$. It is not difficult to see that the same relation holds for all balls $B(x,R)$ in Γ with $R\geq 1$, that is

$$|B(x,R)|\simeq R^\alpha. \tag{3.3}$$

From (3.2) and (3.3), one easily obtains, for all $x\in\Gamma$ and $R\geq 1$,

$$V(x,R)=\mu\left(B(x,R)\right)\simeq R^\alpha\left(R+d\left(x,o\right)\right)^\delta, \tag{3.4}$$

which in particular proves (VD) for (Γ, μ).

Due to the tree structure of Γ, it is easy to compute the Green kernel $g^{\Gamma_k}(x_k, \cdot)$ (cf. [6, Section 4]). Let z_k be the only point in $\overline{\Gamma_k} \setminus \Gamma_k$ (see Fig. 2).

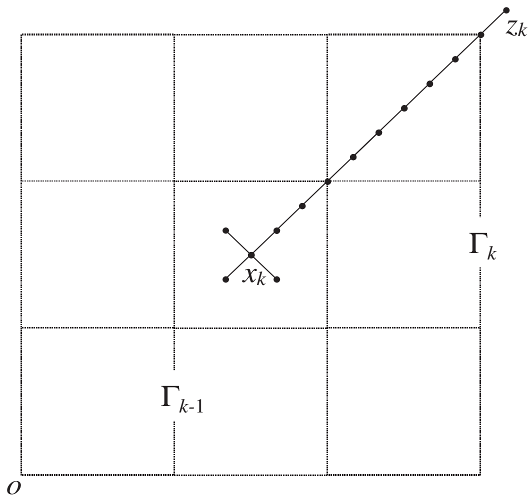

Figure 2 Points x_k and z_k.

Let $g_k(y)$ be a function on Γ satisfying the following conditions:

- g_k vanishes outside Γ_k, in particular at z_k;
- g_k increases linearly along the path from z_k to x_k with the constant increment c_k at each step;
- g_k remains constant along all other paths in Γ_k.

These conditions uniquely determine g_k, up to the choice of c_k. Clearly, g_k is harmonic in $\Gamma_k \setminus x_k$. At point x_k, we have $\Delta g_k = -c_k/4$. Therefore, if we take $c_k := 4/\mu(x_k) = a^{-k}$ then we obtain for all $y \in \Gamma_k$

$$\Delta g_k(y) = -\delta_{x_k}(y)$$

and hence $g_k \equiv g^{\Gamma_k}(x_k, \cdot)$.

For any point y on the paths from z_k to x_k, we have $g_k(y) = c_k d(y, z_k)$. In particular, for any $y \in B(x_k, \frac{1}{3}R_k)$,

$$g_k(y) \simeq \left(\frac{3}{a}\right)^k \quad \text{and} \quad g_k(y)\mu(y) \simeq 3^k.$$

Therefore, by (2.4)

$$E(x_k, R_k) = \sum_{y \in \Gamma_k} g_k(y)\mu(y) \simeq 3^k |\Gamma_k| \simeq 15^k \simeq R_k^\beta$$

where $\beta = \log_3 15$. It is easy to show that the same relation $E(x, R) \simeq R^\beta$ holds for all $x \in \Gamma$ and $R \geq 1$, which proves (E_β).

The Green kernel $g_k = g^{\Gamma_k}(x_k, \cdot)$ constructed above is nearly radial[2]. A similar argument shows that the same is true for all balls in Γ, which implies (H) by [22, Proposition 10.1].

Hence, (Γ, μ) satisfies $(VD) + (H) + (E_\beta)$ and, by Theorem 3.1, (Γ, μ) satisfies $(UE_\beta) + (LE_\beta)$ with the volume function (3.4).

[2]More precisely this means that the Green kernel satisfies a certain condition (HG) described below in Section 4.1.

4 Harnack inequality and resistance

4.1 Harnack inequality for Green function

We say the graph (Γ,μ) satisfies *the ball covering* property (BC) if, for all $\varepsilon>0$, any ball $B(x,R)$ can be covered by $N=N(\varepsilon)$ balls of radii εR. It is well known that (VD) implies (BC).

We say that (Γ,μ) satisfies *the Harnack inequality for the Green function* if, for some (large) constants M,C, for all $x\in\Gamma$ and $R>0$ and for any finite set $U\supset B(x,MR)$,

$$\sup_{y\notin B(x,R)} g^U(x,y)\le C\inf_{z\in B(x,R)} g^U(x,z). \tag{HG}$$

It is possible to show that $(HG)\Longrightarrow(H)$ (cf. [22, Proposition 10.1]). Here we need a converse statement.

Proposition 4.1 *Assume that (Γ,μ) satisfies (p_0). Then $(BC)+(H)\Longrightarrow(HG)$. In particular, $(VD)+(H)\Longrightarrow(HG)$.*

The main part of the proof is contained in the following lemma.

Lemma 4.2 *Assume that (Γ,μ) satisfies (p_0), (BC), (H). Let y,z be two points in a ball $B(x,R)$ such that the shortest path*

$$y=\xi_0\sim\xi_1\sim\ldots\sim\xi_k=z \tag{4.1}$$

connecting y and z in Γ, does not intersect $B(x,\varepsilon R)$ for some $\varepsilon\in(0,1)$. Then, for any finite set U containing $B(x,3R)$,

$$g^U(x,y)\le C_\varepsilon g^U(x,z). \tag{4.2}$$

Proof. If R is in a bounded range then (4.2) follows from the hypothesis (p_0) (cf. Remark 2.1). Assume in the sequel that R is large enough, and observe that for any ξ_i

$$d(x,\xi_i)\le\frac{d(x,y)+d(x,z)+d(y,z)}{2}\le d(x,y)+d(x,z)\le 2R,$$

so that $\xi_i\in B(x,2R)$. By (BC), the ball $B(x,2R)$ can be covered by at most $N(\varepsilon/20)$ balls of radius $r=\frac{\varepsilon}{10}R$. Select out of them only those balls which contain some of ξ_i, and denote their centers by o_j, $j=0,1,...,n$, where $n\le N(\varepsilon/20)$. Clearly, it is possible to enumerate the selected balls so that $B(o_0,r)$ contains y, $B(o_n,r)$ contains z, and $d(o_j,o_{j+1})\le 2r+1$ for all $j<n$. Each ball $B(o_j,r)$ contains at least one of the points ξ_i; select one of them and denote it by $\overline{\xi}_j$. Since $d(x,\overline{\xi}_j)\ge\varepsilon R=10r$, the point x is outside the ball $B(\overline{\xi}_j,10r)$. Since $10r<R$ and hence $B(\overline{\xi}_j,10r)\subset B(x,3R)\subset U$, the function $g^U(x,\cdot)$ is harmonic in $B(\overline{\xi}_j,10r)$. Since $\overline{\xi}_{j+1}\in B(\overline{\xi}_j,5r)$, the Harnack inequality (H) yields $g^U(x,\overline{\xi}_j)\le Cg^U(x,\overline{\xi}_{j+1})$ whence (4.2) follows by iteration. ∎

Proof of Proposition 4.1. The following argument is due to T. Delmotte (private communication). Let y and z be the points where the sup and inf in (HG) are attained, respectively. As follows from the maximum principle, $z\in B(x,R)\setminus B(x,R-1)$ and $y\in B(x,R+1)\setminus B(x,R)$. We need to prove that

$$g^U(x,y)\le Cg^U(x,z) \tag{4.3}$$

where $U\supset B(x,MR)$ for some M. In fact, any $M\ge 14$ will do as we will see below. For a bounded range of R, (4.3) follows from (p_0) (cf. Remark 2.1). Assume in the sequel that R is large enough, and connect y and z by the shortest path in Γ as in (4.1).

CASE 1. All points ξ_i are outside $B(x,\frac{1}{4}R)$. Then (4.3) follows from Lemma 4.2 since $y,z\in B(x,2R)$ and $U\supset B(x,6R)$.

CASE 2. One of the points ξ_i is in the ball $B(x, \frac{1}{4}R)$. Let us show that the shortest path connecting x and z in Γ does not intersect $B(y, \frac{1}{4}R)$. Indeed, denote by ξ the point ξ_i such that $d(x,\xi) < \frac{1}{4}R$. By the triangle inequality, we have

$$d(y,z) = d(y,\xi) + d(z,\xi) \geq (d(x,y) - d(x,\xi)) + (d(x,z) - d(x,\xi)) > d(x,y) + d(x,z) - \frac{R}{2}. \quad (4.4)$$

For any point η on the shortest path between x and z, we obtain by (4.4)

$$d(y,\eta) \geq d(y,z) - d(\eta,z) \geq d(y,z) - d(x,z) \geq d(x,y) - \frac{R}{2} > \frac{R}{4}.$$

Hence, the path from x to z is outside the ball $B(y, \frac{1}{4}R)$. Since $x, z \in B(y, 4R)$ and $U \supset B(y, 12R)$, we obtain by Lemma 4.2 $g^U(y,x) \leq C g^U(y,z)$. Similarly, connecting x to y, we obtain $g^U(z,y) \leq C g^U(z,x)$. Multiplying these inequalities we obtain (4.3). ■

4.2 Green function and resistance

Proposition 4.3 *Assume that the graph (Γ,μ) satisfies (p_0) and (HG). Then for any ball $B(x,R)$ and for any $0 < r \leq R/M$, we have*

$$\sup_{y \notin B(x,r)} g^{B(x,R)}(x,y) \simeq \rho(B(x,r), B(x,R)) \simeq \inf_{y \in B(x,r)} g^{B(x,R)}(x,y). \quad (4.5)$$

Proof. For an arbitrary graph (Γ,μ), the following is true: if A, B are finite subsets of Γ such that $\overline{A} \subset B$ then for any $x \in A^o$

$$\sup_{y \notin A^o} g^B(x,y) \geq \rho(A,B) \geq \inf_{y \in \overline{A}} g^B(x,y), \quad (4.6)$$

where A^o consists of the points in A which have neighbors only in A (see [20, Proposition 4.1] for a continuous version of (4.6)). Applying (4.6) for $A = B(x,r)$, $B = B(x,R)$ and combining with (HG) and (p_0) we obtain (4.5). ■

Proposition 4.4 *Assume that the graph (Γ,μ) satisfies (p_0) and (HG). Fix any ball $B(x,r)$ and denote $B_k = B(x, M^k r)$ for $k = 0, 1, \ldots$. Then for all integers $n > m \geq 0$,*

$$\sup_{y \notin B_m} g^{B_n}(x,y) \simeq \sum_{k=m}^{n-1} \rho(B_k, B_{k+1}) \simeq \inf_{y \in B_m} g^{B_n}(x,y). \quad (4.7)$$

Proof. The following general property of resistance follows directly from the variational definition (2.6):

$$\sum_{k=m}^{n-1} \rho(B_k, B_{k+1}) \leq \rho(B_m, B_n).$$

Together with Proposition 4.3, it implies the lower bound for $\inf g^{B_n}$ in (4.7).

To obtain the upper bound for $\sup g^{B_n}$ observe that the difference $g^{B_{k+1}}(x,\cdot) - g^{B_k}(x,\cdot)$ is a harmonic function in B_k. The maximum principle implies for any $y \in \Gamma$

$$g^{B_{k+1}}(x,y) - g^{B_k}(x,y) \leq \sup_{z \notin B_k} g^{B_{k+1}}(x,z).$$

By Proposition 4.3, we obtain

$$g^{B_{k+1}}(x,y) - g^{B_k}(x,y) \leq C\rho(B_k, B_{k+1}). \quad (4.8)$$

For any $y \notin B_m$, Proposition 4.3 yields

$$g^{B_{m+1}}(x,y) \leq C\rho(B_m, B_{m+1}). \quad (4.9)$$

For such y, adding up (4.9) with (4.8) for $m < k < n$, we obtain the upper bound of $\sup g^{B_n}$ in (4.7). ■

4.3 Proof of $(iii) \Longrightarrow (iv)$ in Theorem 3.1

We need to prove that $(VD)+(H)+(\rho) \Longrightarrow (E)$. Using (2.4), (4.5), (ρ) and (VD), we obtain for $r = R/M$

$$E(x,R) \geq \sum_{y\in B(x,r)} g^{B(x,R)}(x,y)\mu(y) \geq c\rho(B(x,r),B(x,R))V(x,r) \geq cR^{\beta}.$$

For the upper bound, denote $r_k = M^k$, $B_k = B(x,r_k)$ and let n be the minimal integer so that $R < r_n$. Then we have

$$E(x,R) \leq E(x,r_n) = \sum_{y\in B_n} g^{B_n}(x,y)\mu(y) = \sum_{y\in B_0} g^{B_n}(x,y)\mu(y) + \sum_{m=0}^{n-1} \sum_{y\in B_{m+1}\setminus B_m} g^{B_n}(x,y)\mu(y). \tag{4.10}$$

As follows from (p_0) (see Remark 2.1) the first term in the right hand side of (4.10) – the sum over B_0 – is majorized by a multiple of a similar sum over $B_1 \setminus B_0$, which is a part of the second term. Estimating g^{B_n} by Proposition 4.3 and applying (ρ), we obtain

$$\begin{aligned} E(x,R) &\leq C\sum_{m=0}^{n-1}\left[\sum_{k=m}^{n-1}\rho(B_k,B_{k+1})\right]\mu\left(B_{m+1}\setminus B_m\right) \\ &\leq C\sum_{k=0}^{n-1}\left[\sum_{m=0}^{k}\mu\left(B_{m+1}\setminus B_m\right)\right]\rho\left(B_k,B_{k+1}\right) \\ &\leq C\sum_{k=0}^{n-1}\mu(B_{k+1})\rho(B_k,B_{k+1}) \leq C\sum_{k=0}^{n-1} r_{k+1}^{\beta} \leq CR^{\beta}. \end{aligned}$$

5 The resolvent

5.1 Definition of λ, m-resolvent

For any non-empty finite set $B \subset \Gamma$, for all real $\lambda \geq 0$ and integer $m \geq 0$, define the λ, m-resolvent of Δ in B by

$$G_{\lambda,m}^B = \left(\lambda \mathrm{I} - \Delta^B\right)^{-m}. \tag{5.1}$$

Since the spectrum of $-\Delta^B$ is strictly positive, the operator $G_{\lambda,m}^B$ is well-defined. It follows from the definition that $G_{\lambda,0}^B = \mathrm{I}$ and $G_{0,1}^B = G^B$ where G^B is the usual Green function in B. Clearly, for any $m \geq 1$, we have

$$G_{\lambda,1}^B \circ G_{\lambda,m-1}^B = G_{\lambda,m}^B. \tag{5.2}$$

Since $\Delta^B = P^B - \mathrm{I}$, we obtain from (5.1)

$$G_{\lambda,m}^B = \left((\lambda+1)\,\mathrm{I} - P^B\right)^{-m} = \omega^m\left(\mathrm{I} - \omega P^B\right)^{-m}.$$

where $\omega = (\lambda+1)^{-1}$. The binomial formula yields

$$G_{\lambda,m}^B = \omega^m\left(\mathrm{I} + m\omega P^B + \frac{m(m+1)}{2}\omega^2\left(P^B\right)^2 + \ldots\right) = \sum_{n=0}^{\infty} Q_m(n)\omega^{n+m}P_n^B\,, \tag{5.3}$$

where $Q_0(0) = 1$, $Q_0(n) = 0$ for $n \geq 1$, and for $m \geq 1$

$$Q_m(n) = \binom{n+m-1}{m-1} = \frac{(n+m-1)\,(n+m-2)\ldots(n+1)}{(m-1)!}. \tag{5.4}$$

By (5.3) we extend the definition of $G^B_{\lambda,m}$ to infinite sets B. If $\lambda > 0$ then the series in (5.3) always converges. If $\lambda = 0$ then $G^B_{\lambda,m}$ may be equal to ∞ for infinite B.

The λ, m-resolvent has a symmetric kernel defined by

$$g^B_{\lambda,m}(x,y) = \frac{G^B(x,y)}{\mu(y)} = \sum_{n=0}^{\infty} Q_m(n)\omega^{n+m} p^B_n(x,y).$$

If B is finite and $m \geq 1$ then (5.1) implies

$$(\Delta - \lambda)\, g^B_{\lambda,m} = -g^B_{\lambda,m-1} \quad \text{in } B. \tag{5.5}$$

5.2 Upper bound for $0, m$-resolvent

We say that the *mean-value property* (MV) holds on (Γ, μ) if, for any ball $B(x,r)$ and any non-negative harmonic function u in $B(x,r)$,

$$u(x) \leq \frac{C}{V(x,r)} \sum_{y \in B(x,r)} u(y)\mu(y)\,. \tag{MV}$$

Clearly, $(H) + (VD) \Longrightarrow (MV)$ because

$$u(x) \leq C \inf_{B(x,r/2)} u \leq \frac{C}{V(x,r/2)} \sum_{y \in B(x,r/2)} u(y)\mu(y).$$

Lemma 5.1 *If (Γ, μ) satisfies $(VD) + (MV)$ then for all $x \in \Gamma$, $R > 0$, and $y \neq x$,*

$$g^{B(x,R)}(x,y) \leq C \frac{\overline{E}(x,2R)}{V(x,d)}, \tag{5.6}$$

where $d = d(x,y)$.

Proof. If $d > R$ then $g^{B(x,R)}(x,y) = 0$ and there is nothing to prove. Otherwise, consider the function $u(z) = g^{B(x,2R)}(x,z)$. This function is non-negative and harmonic in the ball $B(y,d) \subset B(x,2R)$. Hence, by (MV)

$$u(y) \leq \frac{C}{V(y,d)} \sum_{z \in B(y,d)} u(z)\mu(z) \leq \frac{C}{V(x,d)} \overline{E}(x,2R).$$

Finally, (5.6) follows from $g^{B(x,R)} \leq g^{B(x,2R)}$. ■

Given a set $B \subset \Gamma$, let us endow the space $c_0(B)$ with the sup-norm. Then, regarding G^B as an operator in this space, we have

$$\|G^B\| \leq \overline{E}(B) \tag{5.7}$$

where $\overline{E}$ is defined in (2.5). Indeed, for any $f \in c_0(B)$ and any $x \in B$, (2.4) implies

$$G^B f(x) = \sum_y G^B(x,y) f(y) \leq \sum_y G^B(x,y) \|f\| = E_x(B)\|f\|,$$

whence (5.7) follows by taking sup in x. Iterating (5.7), we obtain

$$\|G^B_{\lambda,m}\| \leq \|G^B_{0,m}\| \leq \overline{E}(B)^m. \tag{5.8}$$

This estimate will be used in the following lemma.

Lemma 5.2 *If (Γ,μ) satisfies $(VD)+(MV)$ then for any $m\geq 1$, for all $x\in\Gamma$, $R>0$, and $y\neq x$,*

$$g_{0,m}^{B(x,R)}(x,y)\leq C\frac{\overline{E}(x,5R)^m}{V(x,d)}, \tag{5.9}$$

where $d=d(x,y)$ and the constant C depends on m and on the constants from the hypotheses.

Proof. The case $m=1$ follows from Lemma 5.1, so we can assume $m\geq 2$ and argue by induction in m. Denote for simplicity $B=B(x,R)$, $G_m^B:=G_{0,m}^B$ and $g_m^B:=g_{0,m}^B$. Assuming $y\in B(x,R)$, let us set $r=d(x,y)/2$ and observe that the balls $B(x,r)$ and $B(y,r)$ do not intersect. Therefore, by (5.2),

$$g_m^B(x,y)=\sum_z g_{m-1}^B(x,z)g_1^B(z,y)\mu(z)\leq\left(\sum_{z\notin B(x,r)}+\sum_{z\notin B(y,r)}\right)g_{m-1}^B(x,z)g_1^B(z,y)\mu(z).$$

Denoting

$$f(z):=g_1^B(z,y)\mathbf{1}_{\{z\notin B(y,r)\}}\quad\text{and}\quad h(z):=g_{m-1}^B(x,z)\,\mathbf{1}_{\{z\notin B(x,r)\}}$$

we obtain

$$g_m^B(x,y)\leq G_{m-1}^Bf(x)+G_1^Bh(y). \tag{5.10}$$

Since the Green kernels in questions are symmetric and increase with B, we obtain by (5.6) and (VD),

$$\|f\|=\sup_{z\notin B(y,r)}g_1^{B(x,R)}(z,y)\leq\sup_{z\notin B(y,r)}g_1^{B(y,2R)}(y,z)\leq C\frac{\overline{E}(y,4R)}{V(y,r)}\leq C\frac{\overline{E}(x,5R)}{V(x,r)},$$

and by the inductive hypothesis

$$\|h\|=\sup_{z\notin B(x,r)}g_{m-1}^{B(x,R)}(x,z)\leq C\frac{\overline{E}(x,5R)^{m-1}}{V(x,r)}.$$

Combining together (5.10), (5.8), and the above estimates for $\|f\|$,$\|g\|$, we obtain (5.9). ■

5.3 Feynman-Kac formula for polyharmonic functions

Let f be a function on Γ such that

$$\Delta f-\lambda f=0\quad\text{in }B, \tag{5.11}$$

for a finite non-empty set $B\subset\Gamma$. Then the function $v=G_{\lambda,m}^Bf$ satisfies

$$(\Delta-\lambda)^{m+1}v=0\quad\text{in }B \tag{5.12}$$

so that v_m is a λ-*polyharmonic function.* Moreover, v satisfies the following boundary conditions outside B:

$$v=0,\ (\Delta-\lambda)\,v=0,\ (\Delta-\lambda)^2\,v=0,\ \ldots,\ (\Delta-\lambda)^m\,v=(-)^m f, \tag{5.13}$$

so that v can be regarded as a solution to the boundary value problem (5.12)-(5.13). The following statement provides a probabilistic representation of such a solution.

Lemma 5.3 (A Feynman-Kac formula) *Let f be a function on Γ satisfying (5.11). Then for any $x\in B$,*

$$f(x)=\mathbb{E}_x\left[\omega^Tf(X_T)\right] \tag{5.14}$$

and, furthermore, for any $m\geq 0$,

$$G_{\lambda,m}^Bf(x)=\mathbb{E}_x\left[Q_{m+1}(T)\omega^{T+m}f(X_T)\right], \tag{5.15}$$

where $T=T_B$ is the first exit time from B.

Proof. For any integer $m \geq -1$, denote

$$v_m(x) = \mathbb{E}_x\left[Q_{m+1}(T)\omega^{T+m} f(X_T)\right]. \tag{5.16}$$

Clearly, $v_{-1} = 0$ in B, $v_0 = f$ in B^c and $v_m = 0$ in B^c for all $m \geq 1$. Let us prove that, for all $m \geq 0$,

$$\Delta v_m - \lambda v_m = -v_{m-1} \quad \text{in } B. \tag{5.17}$$

Indeed, for any $x \in B$, the Markov property implies

$$\mathbb{E}_x\left[Q_{m+1}(T-1)\omega^{T-1+m} f(X_T)\right] = \sum_{y \sim x} P(x,y)\mathbb{E}_y\left[Q_{m+1}(T)\omega^{T+m} f(X_T)\right] = P v_m(y). \tag{5.18}$$

By the property of the binomial coefficients, we have for all $m \geq 0$ and $T \geq 1$

$$Q_{m+1}(T-1) = Q_{m+1}(T) - Q_m(T).$$

Hence, the left hand side of (5.18) is equal to

$$\mathbb{E}_x\left[Q_{m+1}(T)\omega^{T-1+m} f(X_T)\right] - \mathbb{E}_x\left[Q_m(T)\omega^{T-1+m} f(X_T)\right] = \omega^{-1} v_m(x) - v_{m-1}(x).$$

Substituting this into (5.18) and using $\omega^{-1} = 1 + \lambda$, we obtain (5.17).

For $m = 0$ we obtain from (5.17) $\Delta v_0 - \lambda v_0 = 0$. Since $v_0 = f$ outside B, we conclude $v_0 = f$ also in B. Therefore, (5.14) follows from (5.16) for $m = 0$. If $m \geq 1$ then solving (5.17) with the boundary condition $v_m = 0$ outside B, we obtain $v_m = G_\lambda^B v_{m-1}$. Therefore $v_m = G_{\lambda,m}^B f$, whence (5.15) follows. ■

Corollary 5.4 *For any non-empty finite set $B \subset \Gamma$ and for any non-negative function f in Γ such that $\Delta f - \lambda f = 0$ in B,*

$$\left(G_{\lambda,m}^B f(x)\right)^2 \leq c_m f(x) G_{\lambda,2m}^B f(x), \tag{5.19}$$

for all $x \in \Gamma$ and for all $\lambda \geq 0$, $m \geq 0$.

Proof. Using notation (5.16) we have by the Cauchy-Schwarz inequality

$$v_m(x)^2 \leq \mathbb{E}_x\left[\omega^T f(X_T)\right]\mathbb{E}_x\left[Q_{m+1}^2(T)\omega^{T+2m} f(X_T)\right]. \tag{5.20}$$

By (5.4) we obtain $Q_{m+1}^2(T) \leq c_m Q_{2m+1}(T)$. Hence, (5.20) implies $v_m^2 \leq c_m v_0 v_{2m}$ which by Lemma 5.3 coincides with (5.19). ■

Lemma 5.5 ([22, Lemma 7.4]) *Assume that (Γ,μ) satisfies (E). Let f be a non-negative function on Γ, which satisfies in a ball $B = B(x,R)$ the equation $\Delta f - \lambda f = 0$ where $\lambda \in (0,1)$. Then*

$$f(x) \leq C\exp\left(-c\lambda^{1/\beta}R\right)\sup_{B^c} f, \tag{5.21}$$

where the constant $C, c > 0$ depend on (E).

Remark 5.1 The proof in [22] requires an additional restriction $\lambda \geq aR^{-\beta}$ for some $a > 0$. However, if $\lambda < aR^{-\beta}$ then $\lambda^{1/\beta}R$ is bounded and (5.21) holds trivially by adjusting the constant C, since by the maximum principle $f(x) \leq \sup_{B^c} f$.

Corollary 5.6 *Referring to Lemma 5.5, we have for any integer $k \geq 0$*

$$G_{\lambda,k}^B f(x) \leq CR^{\beta k}\exp\left(-c\lambda^{1/\beta}R\right)\sup_{B^c} f. \tag{5.22}$$

Proof. Indeed, by Corollary 5.4 and (5.8), we have

$$\left[G_{\lambda,k}^B f(x)\right]^2 \leq Cf(x)G_{\lambda,2k}^B f(x) \leq Cf(x)\|G_{\lambda,2k}^B\|\sup_B f \leq Cf(x)\overline{E}(B)^{2k}\sup_{B^c} f.$$

Using the estimate (5.21) for $f(x)$ and (E), we conclude the proof. ■

5.4 Upper bound for λ, m-resolvent

Theorem 5.7 *Assume that* (Γ, μ) *satisfies* $(VD) + (MV) + (E)$. *Then for a large enough* $m > 1$ *and for all* $0 < \lambda < 1$, $x \in \Gamma$

$$g_{\lambda,m}(x,x) \le C \frac{\lambda^{-m}}{V(x, \lambda^{-1/\beta})}. \tag{G}$$

We start with a lemma.

Lemma 5.8 *Assume that* (Γ, μ) *satisfies* $(VD) + (MV) + (E)$. *Then for all* $x \in \Gamma$, $0 < r < R$

$$g_{\lambda,m}^{B(x,R)}(x,x) - g_{\lambda,m}^{B(x,r)}(x,x) \le C \frac{R^{\beta m}}{V(x,r)} \exp\left(-c\lambda^{1/\beta} r\right).$$

Proof. Denote for simplicity $A = B(x,R)$, $B = B(x,r)$ and set

$$v_m(y) := g_{\lambda,m}^A(x,y) - g_{\lambda,m}^B(x,y).$$

Clearly, we have $v_0 = 0$ and $\Delta v_m - \lambda v_m = -v_{m-1}$ in B for $m \ge 1$. Therefore,

$$v_m = G_\lambda^B v_{m-1} + u_m \quad \text{in } B, \tag{5.23}$$

where u_m is defined by

$$\begin{cases} \Delta u_m - \lambda u_m = 0 & \text{in } B, \\ u_m|_{B^c} = v_m. \end{cases}$$

Iterating (5.23) and using $v_0 = 0$ we obtain

$$v_m = G_{\lambda,m-1}^B u_1 + G_{\lambda,m-2}^B u_2 + \dots + u_m. \tag{5.24}$$

By Corollary 5.6

$$G_{\lambda,l}^B u_k(x) \le C r^{\beta l} \exp\left(-c\lambda^{1/\beta} r\right) \sup_{A \setminus B} v_k, \tag{5.25}$$

and by Lemma 5.2

$$\sup_{A\setminus B} v_k = \sup_{y \in A \setminus B} g_{\lambda,k}^A(x,y) \le C \frac{R^{\beta k}}{V(x,r)}. \tag{5.26}$$

Therefore, we obtain from (5.24), (5.25), and (5.26)

$$v_m(x) = \sum_{k=1}^m G_{\lambda,m-k}^B u_k(x) \le C \sum_{k=1}^m r^{\beta(m-k)} \exp\left(-c\lambda^{1/\beta} r\right) \sup_{A\setminus B} v_k \le C \frac{R^{\beta m}}{V(x,r)} \exp\left(-c\lambda^{1/\beta} r\right),$$

which was to be proved. ∎

Proof of Theorem 5.7. Denote $R_k = 2^k$ and $B_k = B(x, 2^k)$. We have

$$g_{\lambda,m}(x,x) = g_{\lambda,m}^{B_0}(x,x) + \sum_{k=0}^\infty \left(g_{\lambda,m}^{B_{k+1}}(x,x) - g_{\lambda,m}^{B_k}(x,x)\right).$$

The first term $g_{\lambda,m}^{B_0}(x,x)$ is estimated as follows:

$$g_{\lambda,m}^{B_0}(x,x) = \sum_{n=0}^\infty Q_m(n) \omega^{n+m} P_n^{B(x,1)}(x,x) = Q_m(0)\omega^m P_0^{B(x,1)}(x,x) = \omega^m \le 1.$$

Using Lemma 5.8 to estimate the other terms, we obtain

$$g_{\lambda,m}(x,x) \le 1 + C\sum_{k=0}^{\infty} \exp\left(-c\lambda R_k^{1/\beta}\right) \frac{R_{k+1}^{\beta m}}{V(x,R_k)}.$$

Denote $R = \lambda^{-1/\beta}$ and rewrite this inequality as follows

$$g_{\lambda,m}(x,x) \le 1 + C\sum_{k=0}^{\infty} \exp\left[-c\left(\frac{2^k}{R}\right)^{1/\beta}\right]\left(\frac{2^k}{R}\right)^{\beta m} \frac{V(x,R)}{V(x,2^k)}\frac{R^{\beta m}}{V(x,R)}. \tag{5.27}$$

If $2^k \le R$ then by (2.1)

$$\frac{V(x,R)}{V(x,2^k)} \le C\left(\frac{R}{2^k}\right)^{\alpha}. \tag{5.28}$$

Let us choose m so large that $\beta m > \alpha$. If $2^k \le R$ then the k-th term in the sum (5.27) is estimated from above by

$$\left(\frac{2^k}{R}\right)^{\beta m-\alpha} \frac{R^{\beta m}}{V(x,R)}.$$

Since this is an increasing geometric series, the sum of all such terms over all k such that $2^k \le R$, is bounded by a multiple of the largest term, that is by

$$C\frac{R^{\beta m}}{V(x,R)}. \tag{5.29}$$

If $2^k > R$ then the k-th term in (5.27) is bounded by

$$\exp\left[-c\left(\frac{2^k}{R}\right)^{1/\beta}\right]\left(\frac{2^k}{R}\right)^{\beta m} \frac{R^{\beta m}}{V(x,R)} \le C\exp\left[-\frac{c}{2}\left(\frac{2^k}{R}\right)^{1/\beta}\right]\frac{R^{\beta m}}{V(x,R)}.$$

Since this decreases faster than a geometric series, the sum of such terms over all k such that $2^k > R$ is again bounded by (5.29). Finally, by (5.28) with $k=0$, we have $V(x,R) \le CR^{\alpha} \le CR^{\beta m}$ which means that the term 1 in (5.27) is also bounded by (5.29). Hence, $g_{\lambda,m}(x,x)$ is bounded by (5.29) whence (G) follows by substituting $R = \lambda^{-1/\beta}$. ∎

6 Estimates of the heat kernel

6.1 Diagonal upper estimate

Theorem 6.1 *If (Γ,μ) satisfies $(VD)+(G)$ then, for all $x,y \in \Gamma$ and $n \ge 1$,*

$$p_n(x,x) \le \frac{C}{V(x,n^{1/\beta})} \tag{DUE}$$

and

$$p_n(x,y) \le \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}}. \tag{PUE}$$

Proof. Let us first prove that, for $\lambda = n^{-1}$ and any $m \ge 0$,

$$p_{2n}(x,x) \le C\lambda^m g_{\lambda,m}(x,x). \tag{6.1}$$

Indeed, $p_{2k}(x,x)$ is non-increasing in k. Therefore, for $\lambda = n^{-1}$ we have

$$\begin{aligned} g_{\lambda,m}(x,x) &= \sum_{k=0}^{\infty} Q_m(k)\omega^{k+m} p_k(x,x) \geq \sum_{k=0}^{\infty} Q_m(2k)\omega^{2k+m} p_{2k}(x,x) \\ &\geq c\sum_{k=1}^{n} k^{m-1}\omega^{2k+m} p_{2k}(x,x) \geq cn^m p_{2n}(x,x), \end{aligned}$$

(where we have used $\omega^{2k+m} \geq (1+1/n)^{-2n-m} \geq e^{-2}2^{-m}$) whence (6.1) follows. By the hypothesis (G) we have for some m

$$g_{\lambda,m}(x,x) \leq \frac{C}{\lambda^m V(x,\lambda^{-1/\beta})},$$

which together with (6.1) implies (DUE) for even n.

Using the semigroup property and the Cauchy-Schwarz inequality, we obtain (PUE) for even n:

$$\begin{aligned} p_{2n}(x,y) &= \sum_z p_n(x,z)p_n(z,y)\mu(z) && (6.2) \\ &\leq \sqrt{p_{2n}(x,x)p_{2n}(y,y)} && (6.3) \\ &\leq \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}}. \end{aligned}$$

Again by the semigroup property, we have

$$p_{2n+1}(x,y) = \sum_{z\sim x} P_1(x,z)p_{2n}(z,y) \leq \sup_{z\sim x} p_{2n}(z,y) \tag{6.4}$$

whence

$$p_{2n+1}(x,y) \leq \sup_{z\sim x} p_{2n}(z,y) \leq \sup_{z\sim x} \frac{C}{\sqrt{V(z,n^{1/\beta})V(y,n^{1/\beta})}}.$$

By (VD), $V(z,n^{1/\beta}) \simeq V(x,n^{1/\beta})$ for all $z\sim x$, whence (PUE) and (DUE) follow for odd $n>1$. Finally, (PUE) and (DUE) for $n=1$ follow directly from the definition (2.2) of $p_n(x,y)$. ∎

6.2 Off-diagonal upper estimate

Theorem 6.2 *For any graph* (Γ,μ)

$$(VD)+(DUE)+(E) \Longrightarrow (UE).$$

The following statement is a simple consequence of [22, Lemma 7.4] (the latter was proved in [22] using Lemma 5.5 cited above).

Lemma 6.3 *Let* (Γ,μ) *satisfy* (E) *and let* f *be a bounded non-negative function on* Γ *which vanishes in some ball* $B(x,R)$*. Then for any* $n\geq 1$ *the function* $P_n f$ *admits the following estimate at the point* x*:*

$$P_n f(x) \leq C\exp\left[-c\left(\frac{R^\beta}{n}\right)^{\frac{1}{\beta-1}}\right]\sup f. \tag{6.5}$$

Proof of Theorem 6.2. We can assume that (PUE) holds because, as was shown in the proof of Theorem 6.1, (PUE) follows from $(VD)+(DUE)$.

STEP 1. Let us prove that $(PUE)+(VD)+(E)$ implies, for all $R \geq 0$,

$$I_n(x,R) := \sum_{z \notin B(x,R)} p_n^2(x,z)\mu(z) \leq \frac{C}{V(x,n^{1/\beta})} \exp\left[-\left(\frac{R^\beta}{Cn}\right)^{\frac{1}{\beta-1}}\right]. \tag{6.6}$$

By (2.1) we have for all $x,y \in \Gamma$, $n>0$, $\varepsilon>0$

$$\frac{V(x,n^{1/\beta})}{V(y,n^{1/\beta})} \leq C\left(1+\frac{d(x,y)}{n^{1/\beta}}\right)^\alpha \leq C_\varepsilon \exp\left[\varepsilon\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{6.7}$$

Therefore, by (PUE) and (6.7),

$$p_n(x,y) \leq \frac{C}{V(x,n^{1/\beta})}\left[\frac{V(x,n^{1/\beta})}{V(y,n^{1/\beta})}\right]^{1/2} \leq \frac{C_\varepsilon}{V(x,n^{1/\beta})} \exp\left[\varepsilon\left(\frac{d^\beta(x,y)}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{6.8}$$

If $R^\beta \leq Cn$ then (6.6) follows from (6.3) and (DUE). Assuming in the sequel $R^\beta > Cn$, denote $R_k = 2^k R$ where $k=0,1,2,\ldots$. Splitting the summation in (6.6) to annuli $A_k := B(x,R_k)\setminus B(x,R_{k-1})$, denoting $f_k = \mathbf{1}_{A_k} p_n(x,\cdot)$, and using Lemma 6.3, we obtain

$$I_n(x,R) = \sum_{k=1}^\infty \sum_{z\in A_k} p_n^2(x,z)\mu(z) = \sum_{k=1}^\infty P_n f_k(x) \leq C\sum_{k=1}^\infty \exp\left[-c\left(\frac{R_{k-1}^\beta}{n}\right)^{\frac{1}{\beta-1}}\right] \sup f_k\,. \tag{6.9}$$

By (6.8), we obtain

$$\sup f_k \leq \sup_{y\notin B(x,R_k)} p_n(x,y) \leq \frac{C_\varepsilon}{V(x,n^{1/\beta})} \exp\left[\varepsilon\left(\frac{R_k^\beta}{n}\right)^{\frac{1}{\beta-1}}\right].$$

Substituting this estimate in (6.9) and estimating the sum in (6.9) for small enough ε, we obtain (6.6).

STEP 2. Let us deduce (UE) from (6.6). Denote $R = \frac{1}{2}d(x,y)$ and observe that Γ is covered by the union of $B(x,R)^c$ and $B(y,R)^c$. Therefore, we have by (6.2)

$$p_{2n}(x,y) \leq \sum_{z\notin B(x,R)} p_n(x,z)p_n(z,y)\mu(z) + \sum_{z\notin B(y,R)} p_n(x,z)p_n(z,y)\mu(z). \tag{6.10}$$

By the Cauchy-Schwarz inequality, the first sum in (6.10) is dominated by $\sqrt{I_n(x,R)I_n(y,0)}$, and the second sum is dominated by $\sqrt{I_n(y,R)I_n(x,0)}$. Applying (6.6) we obtain

$$p_{2n}(x,y) \leq \frac{C}{\sqrt{V(x,n^{1/\beta})V(y,n^{1/\beta})}} \exp\left[-\left(\frac{d^\beta(x,y)}{Cn}\right)^{\frac{1}{\beta-1}}\right],$$

which together with (6.7) yields (UE) for even n. Finally, (UE) for odd n follows from (6.4). ■

6.3 Proof of $(iv) \Longrightarrow (i)$ in Theorem 3.1

Let us prove that $(VD)+(H)+(E) \Longrightarrow (UE)$. As was shown in Section 5.2 $(VD)+(H) \Longrightarrow (MV)$. By Theorem 5.7 $(VD)+(MV)+(E) \Longrightarrow (G)$ and by Theorem 6.1 $(VD)+(G) \Longrightarrow (DUE)$. By Theorem 6.2 $(VD)+(DUE)+(E) \Longrightarrow (UE)$, which finishes the proof.

Remark 6.1 Clearly, the hypothesis (H) can be replaced here by (MV) so that we obtain (3.1). Note also that the lower bound in the hypothesis (E) was required only for Lemma 5.5 and for its consequence Lemma 6.3. In all other places we used only the upper bound $E(x,R) \le CR^\beta$.

Let us prove that $(VD) + (H) + (E) \Longrightarrow (LE)$. By [22, Propositions 7.1,9.1] $(VD) + (E)$ implies

$$p_{2n}^{B(x,R)}(x,x) \ge \frac{c}{V(x,n^{1/\beta})}, \quad \text{for all } n \le \varepsilon R^\beta, \tag{6.11}$$

for some $c, \varepsilon > 0$. By Theorem 6.1 $(VD) + (H) + (E) \Longrightarrow (DUE)$. By [22, Proposition 13.1], [33], assuming (p_0), the conditions

$$(VD) + (H) + (E) + (DUE) + (6.11)$$

imply

$$p_n(x,y) + p_{n+1}(x,y) \ge \frac{c}{V(x,n^{1/\beta})} \quad \text{whenever } d(x,y) \le \delta n^{1/\beta}, \tag{6.12}$$

for some $\delta > 0$. Finally, using (p_0) and arguing as in [22, Proposition 13.2] or [12, Theorem 3.8] or [33], we obtain (VD)+(6.12)$\Longrightarrow (LE)$.

Remark 6.2 Alternatively, one can deduce (LE) from $(VD) + (H) + (UE)$ using a modification of the method described in [22, Remark 15.1] (in the continuous setting, this was done in [23]). In particular, we see that $(VD) + (H) + (UE)$ is equivalent to each of the conditions $(i) - (iv)$ of Theorem 3.1.

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