Upper bounds for transition probabilities on graphs and isoperimetric inequalities

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Abstract

In this paper necessary and sufficient conditions are presented for heat kernel upper bounds for random walks on weighted graphs. Several equivalent conditions are given in the form of isoperimetric inequalities.

Contents

1 Introduction

Heat kernel upper bounds are subject of heavy investigations for decades. Aronson, Moser, Varopoulos, Davies and many more contributed to the development of the area (for the history see the bibliography of [22]). The work of Varopoulos highlighted the connection between the heat kernel upper estimates and isoperimetric inequalities. The present paper follows this approach and provides transition probability upper estimates of reversible Markov chains in a general form under necessary and sufficient conditions. The conditions are isoperimetric inequalities which control the spectral gap, the capacity or the mean exit time of a finite vertex set. In addition the paper present a generalization of the Davies-Gaffney inequality (c.f. [4]) which is a tool in the proof of the off-diagonal upper estimate.

Let us consider a countable infinite connected graph Γ.

Definition 1.1 *A symmetric weight function $\mu_{x,y} = \mu_{y,x} > 0$ is given on the edges $x \sim y$. This weight function induces a measure $\mu(x)$*

$$\begin{aligned} \mu(x) &= \sum_{y \sim x} \mu_{x,y}, \\ \mu(A) &= \sum_{y \in A} \mu(y). \end{aligned}$$

The graph is equipped with the usual (shortest path length) graph distance $d(x,y)$ and open metric balls are defined for $x \in \Gamma$, $R > 0$ as

$$B(x,R) = \{y \in \Gamma : d(x,y) < R\}$$

and its $\mu-$measure is denoted by $V(x,R)$

$$V(x,R) = \mu(B(x,R)).$$

The weighted graph has the volume doubling property (**VD**) *if there is a constant $D_V > 0$ such that for all $x \in \Gamma$ and $R > 0$*

$$V(x,2R) \leq D_V V(x,R). \tag{1.1}$$

Definition 1.2 *The edge weights define a reversible Markov chain $X_n \in \Gamma$, i.e. a random walk on the weighted graph (Γ, μ) with transition probabilities*

$$\begin{aligned} P(x,y) &= \frac{\mu_{x,y}}{\mu(x)}, \\ P_n(x,y) &= \mathbb{P}(X_n = y | X_0 = x). \end{aligned}$$

The "heat kernel" of the random walk is

$$p_n(x,y) = p_n(y,x) = \frac{1}{\mu(y)} P_n(x,y).$$

Let $\mathbb{P}_x, \mathbb{E}_x$ denote the probability measure and expected value with respect to the Markov chain X_n if $X_0 = x$.

Definition 1.3 *The Markov operator P of the reversible Markov chain is naturally defined by*

$$Pf(x) = \sum P(x,y) f(y).$$

Definition 1.4 *The Laplace operator on the weighted graph (Γ, μ) is defined simply as*

$$\Delta = P - I.$$

Definition 1.5 *For $A \subset \Gamma$ consider P^A the Markov operator P restricted to A. This operator is the Markov operator of the killed Markov chain, which is killed on leaving A, also corresponds to the Dirichlet boundary condition on A. Its iterates are denoted by P_k^A.*

Definition 1.6 *The Laplace operator with Dirichlet boundary conditions on a finite set $A \subset \Gamma$ defined as*

$$\Delta^A f(x) = \begin{cases} \Delta f(x) & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}.$$

The smallest eigenvalue of $-\Delta^A$ is denoted in general by $\lambda(A)$ and for $A = B(x,R)$ it is denoted by $\lambda = \lambda(x,R) = \lambda(B(x,R))$.

Definition 1.7 *On the weighted graph (Γ, μ) the inner product is defined as*

$$(f,g) = (f,g)_\mu = \sum_{x \in \Gamma} f(x) g(x) \mu(x).$$

Definition 1.8 *The energy or Dirichlet form $\mathcal{E}(f,f)$ associated to the Laplace operator Δ is defined as*

$$\mathcal{E}(f,f) = -(\Delta f, f) = \frac{1}{2} \sum_{x,y \in \Gamma} \mu_{x,y} (f(x) - f(y))^2.$$

Using this notation the smallest eigenvalue of $-\Delta^A$ can be defined by

$$\lambda(A) = \inf\left\{\frac{\mathcal{E}(f,f)}{(f,f)} : f \in c_0(A), f \neq 0\right\} \tag{1.2}$$

as well.

The exit time from a set $A \subset \Gamma$ is

$$T_A = \min\{k \geq 0 : X_k \in \Gamma \backslash A\}$$

and its expected value is denoted by

$$E_x(A) = \mathbb{E}(T_A | X_0 = x)$$

and we will use the $E = E(x,R) = E_x(x,R) = E_x(B(x,R))$ short notations.

In the whole sequel c, C will denote unimportant constants, their values may change from place to place.

Notation 1 *For two real series $a_\xi, b_\xi, \xi \in S$ we shall use the notation $a_\xi \simeq b_\xi$ if there is a $C \geq 1$ such that for all $\xi \in S$*

$$\frac{1}{C} a_\xi \leq b_\xi \leq C a_\xi.$$

The main concerns of this paper are upper estimates of the heat kernel. Such estimates have a wast literature, (see the bibliography of [5] as a starting point).

The diagonal upper estimate

$$p_n(x,x) \leq C n^{-\gamma}$$

is equivalent with the Faber-Krahn inequality

$$\lambda^{-1}(A) \leq C\mu(A)^\delta \text{ for all } A \subset \Gamma$$

for some $\gamma, \delta, C > 0$ (c.f. [2],[8]).

The classical off-diagonal upper estimate has the form

$$p_n(x,x) \leq \frac{C_d}{n^{\frac{d}{2}}} \exp\left[-\frac{d^2(x,y)}{2n}\right]$$

for the random walk on the integer lattice $\mathbb{Z}^d$ which reflects the basic fact that

$$E(x,R) \simeq R^2.$$

Coulhon and Grigor'yan [4] proved for random walks on weighted graphs that the relative Faber-Krahn inequality

$$\lambda^{-1}(A) \leq CR^2 \left(\frac{\mu(A)}{V(x,R)} \right)^{\delta} \text{ for all } A \subset B(x,R), x \in \Gamma, R > 0$$

is equivalent with the conjunction of the volume doubling property (VD) and

$$p_n(x,y) \leq \frac{C}{V(x,\sqrt{n})} \exp\left[-\frac{d^2(x,y)}{Cn} \right].$$

In the last fifteen years several works were devoted to the study of sub-diffusive behavior of fractals, which typically means that the condition (E_β)

$$E(x,R) \simeq R^\beta \tag{1.3}$$

for a $\beta > 2$ is satisfied. On particular fractals it was possible to show that the following a heat kernels upper bound:

$$p_t(x,y) \leq \frac{C}{V\left(x, t^{\frac{1}{\beta}}\right)} \exp\left[-\left(\frac{R^\beta}{Ct} \right)^{\frac{1}{\beta-1}} \right] \tag{UE_β}$$

holds. Grigor'yan have shown in [9] that in contiuous settings under the volume doubling condition (UE_β) is equivalent with the conjunction of (E_β) and

$$\lambda^{-1}(A) \leq CR^\beta \left(\frac{\mu(A)}{V(x,R)} \right)^{\delta} \text{ for all } A \subset \Gamma, x \in \Gamma, R > 0,$$

The upper estimate (UE_β) has been shown for several particular fractals prior to [9] (see the literature in [14] or for very recent ones in [9],[13],[11] or [21]) and generalized to some class of graphs in [19] and [21]. In [11] an example is given for a graph which satisfies (UE_β) and the lower counterpart (differing only in the constants C, c). This example is an easy modification of the Vicsek tree (See Figure 1.).

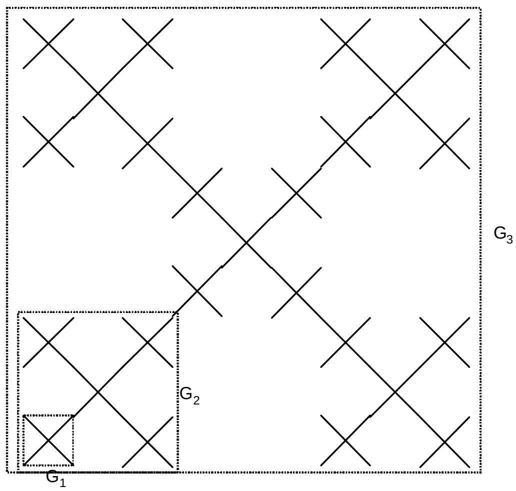

Figure 1

One should put increasing weights on the edges of the increasing blocks of the tree (See Figure 2.).

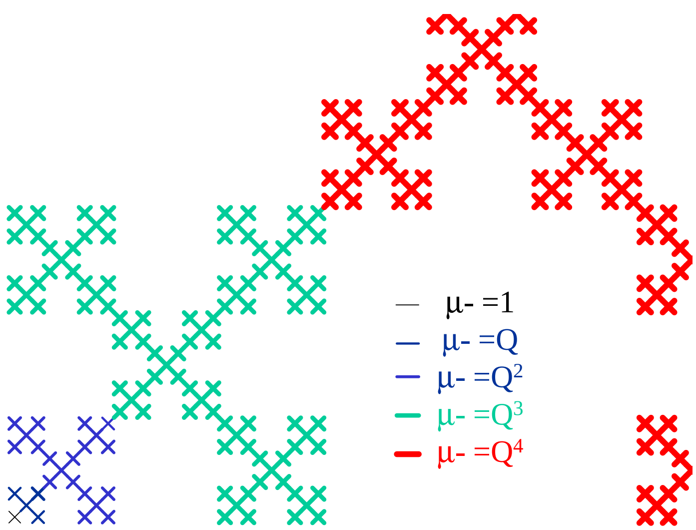

Figure 2

It is easy to see that on this tree the volume doubling condition and (E_β) holds. An other construction based on the Vicsek tree is the strached Vicsek tree which is given in [21] and it violates (E_β) while it satisfies (VD) It can be obtained by replacing the edges of the consequtive block of the tree with paths of slowly increasing lenght.

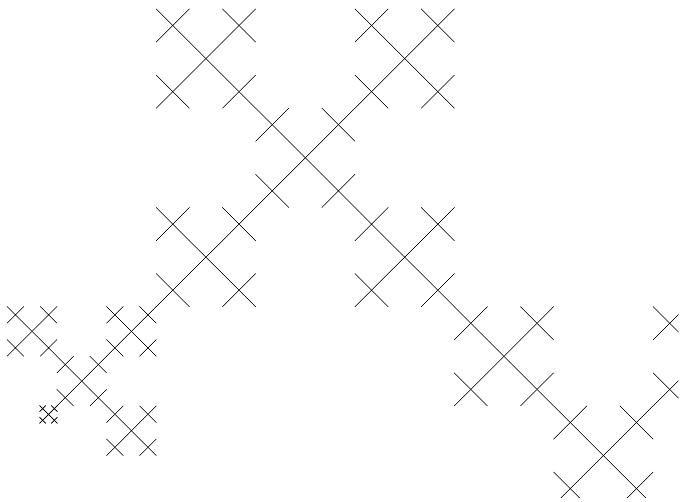

Figure 3

This example is not covered by any earlier results but it was shown in [21] that satisfis enough regularity properties to obtain a heat kernel upper estimate which is local not only in the volume but in the mean exit time as well. We shall return to this example briefly in Section 5.

The main result of the present paper gives equivalent isoperimetric inequalities which imply on- and off-diagonal upper estimates in a general form. Let us give here only one, the others will be stated after the needed definitions.

The result states among others that if there are $C, \delta > 0$ such that for all $x \in \Gamma, R, n > 0$ if for all $A \subset B(x, 3R), B = B(x, R), 2B = B(x, 2R)$

$$\lambda^{-1}(A) \leq CE(x, R)\left(\frac{\mu(A)}{\mu(2B)}\right)^{\delta}$$

holds then, the (local) diagonal upper estimate (DUE) holds: there is a $C > 0$, such that for all $x \in \Gamma, n > 0$

$$p_n(x, x) \leq \frac{C}{V(x, e(x, n))} \tag{DUE}$$

and the (local) upper estimate (UE) holds: there are $c, C > 0, \beta > 1$ such that for all $x, y \in \Gamma, n > 0$

$$p_n(x, y) \leq \frac{C}{V(x, e(x, n))} \exp\left[-c\frac{E(x, d(x, y))^{\frac{1}{\beta-1}}}{n}\right]. \tag{UE}$$

Here $e(x,n)$ is the inverse of $E(x,R)$ in the second variable. The existance follows easily from the strong Markov property (c.f. [20]). The full results contain the corresponding reverse implications as well.

The presented results are motivated by the work of Kigami [13] and Grigor'yan [9]. Those provide necessary and sufficient conditions for the case when $E(x,R) \simeq R^{\beta}$ uniformly over the space (they work in the continuous settings on measure metric spaces). Our result is adaptation to the discrete settings but generalization of the mentioned works relaxing the condition on the mean exit time. It seems that the results carry over to the continuous setup without major changes provided the stohastic process has some natural properties (which among others imply that it has continuous heat kernel, c.f. [9]).

The structure of the paper is the following. In Section 2 we lay down the necessary definitions and give the statement of the main results. In Section 3 some potential theoretic inequalities are collected and equivalence of the isoperimetric inequalities are given. In Section 4 the proof of the main result is given. Finally Section 5 provides further details of the example of the strached Vicsek tree.

2 Basic definitions and the results

We consider the weighted graph (Γ, μ) as it was introduced in the previous section.

Condition 1 *In many statements we assume that condition* $(\mathbf{p}_0)$ *holds, that is, there is an universal* $p_0 > 0$ *such that for all* $x, y \in \Gamma, x \sim y$

$$\frac{\mu_{x,y}}{\mu(x)} \geq p_0, \tag{2.1}$$

Notation 2 *The following standard notations will be used.*

$$\|f\|_1 = \sum_{x\in\Gamma} |f(x)| \mu(x)$$

and

$$\|f\|_2 = (f,f)^{1/2}$$

Definition 2.1 *We introduce*

$$G^A(y,z) = \sum_{k=0}^{\infty} P_k^A(y,z)$$

the local Green function, the Green function of the killed walk and the corresponding Green kernel as

$$g^A(y,z) = \frac{1}{\mu(z)} G^A(y,z).$$

Definition 2.2 *Let ∂A denote the boundary of a set $A \subset \Gamma : \partial A = \{z \in \Gamma \backslash A : z \sim y \in A \}$. The closure of A will be denoted by $\overline{A}$ and defined by $\overline{A} = A \cup \partial A$, also let $A^c = \Gamma \backslash A$.*

For convenience we introduce a short notation for the volume of the annulus $B(x,R) \backslash B(x,r)$ for $R > r > 0$:

$$v(x,r,R) = V(x,R) - V(x,r).$$

Definition 2.3 *The extreme mean exit time is defined as*

$$\overline{E}(A) = \max_{x \in A} E_x(A)$$

and the $\overline{E}(x,R) = \overline{E}(B(x,R))$ simplified notation will be used.

Definition 2.4 *We say that the graph satisfies condition $\left(\overline{E}\right)$ if there is a $C > 0$ such that for all $x \in \Gamma, R > 0$*

$$\overline{E}(x,R) \leq CE(x,R).$$

Definition 2.5 *We will say that the weighted graph (Γ, μ) satisfies the time comparison principle (**TC**) if, here is a constant $C > 1$ such that for all $x \in \Gamma$ and $R > 0, y \in B(x,R)$*

$$\frac{E(y,2R)}{E(x,R)} \leq C. \tag{2.2}$$

Remark 2.1 *It is clear that (TC) implies $\left(\overline{E}\right)$.*

Definition 2.6 *For any two disjoint sets, $A, B \subset \Gamma$, the resistance between them $\rho(A,B)$ is defined as*

$$\rho(A,B) = \left(\inf\left\{\mathcal{E}(f,f) : f|_A = 1, f|_B = 0\right\}\right)^{-1} \tag{2.3}$$

and we introduce

$$\rho(x,S,R) = \rho(B(x,S), \Gamma \backslash B(x,R))$$

for the resistance of the annulus about $x \in \Gamma$, with $R > S \geq 0$.

Theorem 2.1 *Assume that (Γ,μ) satisfies (p_0) then, the following inequalities are equivalent (asuming that each statement separately holds for all $x,y\in\Gamma, R>0, n>0, D\subset A\subset B=B(x,3R)$ with fixed independent $\delta, C>0, \beta>1$)*

$$\overline{E}(A)\leq CE(x,R)\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}, \tag{E}$$

$$\lambda(A)^{-1}\leq CE(x,R)\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}, \tag{FK}$$

$$\rho(D,A)\mu(D)\leq CE(x,R)\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}, \tag{ρ}$$

$$p_n(x,x)\leq\frac{C}{V(x,e(x,n))} \tag{DUE}$$

together with (VD) and (TC),

$$p_n(x,y)\leq\frac{C}{V(x,e(x,n))}\exp\left(-c\left(\frac{E(x,R)}{n}\right)^{\frac{1}{\beta-1}}\right) \tag{UE}$$

together with (VD) and (TC), where $e(x,n)$ is the inverse of $E(x,R)$ in the second variable.

Corollary 2.2 *If (Γ,μ) satisfies $(p_0),(VD)$ and (TC) then the following statements are equivalent.(assuming that each statement seprately holds for all $x,y\in\Gamma, R>0, n>0, D\subset A\subset B=B(x,2R)$ with fixed $\delta, C>0, \beta>1$)*

$$\frac{\overline{E}(A)}{\overline{E}(B)}\leq C\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}, \tag{2.4}$$

$$\frac{\lambda^{-1}(A)}{\lambda^{-1}(B)}\leq C\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}, \tag{2.5}$$

$$\frac{\rho(D,A)}{\rho(x,R,2R)}\leq C\left(\frac{\mu(A)}{\mu(D)}\right)^{\delta}\left(\frac{\mu(D)}{\mu(B)}\right)^{\delta-1}, \tag{2.6}$$

$$p_n(x,x)\leq\frac{C}{V(x,e(x,n))}, \tag{2.7}$$

$$p_n(x,y)\leq\frac{C}{V(x,e(x,n))}\exp\left(-c\left(\frac{E(x,R)}{n}\right)^{\frac{1}{\beta-1}}\right). \tag{2.8}$$

Remark 2.2 *Now let us relate the present results to the conditions given in [21] . Among other equivalent conditions the (elliptic) mean value inequality was used in [21] . It says that for all u nonnegative harmonic functions in* $B(x,R)$

$$u(x) \leq \frac{C}{V(x,R)} \sum_{y \in B(x,R)} u(y) \mu(y). \tag{2.9}$$

It was shown in [21] that under $(p_0)+(VD)+(TC)$ *the mean value inequality is equivalent with the diagonal upper estimate* (DUE). *This means that the mean value inequality is equivalent with the relative isoperimetric inequalities* $(E), (FK), (\rho)$ *and* $(2.4-2.6)$ *provided* (VD) *and* (TC) *holds. In [12] a direct proof of* $(MV) \Longrightarrow (FK)$ *is given for measure metric spaces which works for weighted graphs as well.*

3 Basic inequalities

In this section basic inequalities are collected, several of them are known, some of them are new.

Lemma 3.1 *(cf [4]) If* (p_0) *and* (VD) *hold then for all* $x \in \Gamma, R > 0$, $y \in B(x,R)$ *then*

$$\frac{V(x,R)}{V(y,R)} \leq C, \tag{3.1}$$

furthermore there is an A_V *such that for all* $x \in \Gamma, R > 0$

$$2V(x,R) \leq V(x, A_V R), \tag{3.2}$$

$$V(x, MR) - V(x,R) \simeq V(x,R) \tag{3.3}$$

for any fixed $M \geq 2$, *and there is an* $\alpha > 0$ *such that for all* $y \in B(x,R), S \leq R$

$$\frac{V(x,R)}{V(y,S)} \leq C \left(\frac{R}{S}\right)^{\alpha}. \tag{3.4}$$

The inequality (3.2), sometimes called anti-doubling property. As we already mentioned, (VD) is equivalent with (3.1) and it is again evident that both are equivalent with the inequality

$$\frac{V(x,R)}{V(y,S)} \leq C \left(\frac{R}{S}\right)^{\alpha}, \tag{3.5}$$

where $\alpha = \log_2 D_V$ and $d(x,y) < R$. The next Proposition is taken from [10] (see also [21])

Proposition 3.2 *If* (p_0) *holds, then, for all* $x, y \in \Gamma$ *and* $R > 0$ *and for some* $C > 1$,

$$V(x,R) \leq C^R \mu(x), \tag{3.6}$$

$$p_0^{d(x,y)} \mu(y) \leq \mu(x) \tag{3.7}$$

and for any $x \in \Gamma$

$$|\{y : y \sim x\}| \leq \frac{1}{p_0}. \tag{3.8}$$

Now we recall some results from [20] which connect the mean exit time, the spectral gap, volume and resistance growth.

Theorem 3.3 $(p_0), (VD)$ *and* (TC) *implies that*

$$\lambda^{-1}(x,2R) \asymp E(x,2R) \asymp \overline{E}(x,2R) \asymp \rho(x,R,2R)\, v(x,R,2R). \tag{3.9}$$

Theorem 3.4 *For a weighted graph* (Γ, μ) *if*

$$\frac{E(x,R)}{E(y,R)} \leq C \tag{3.10}$$

for all $x \in \Gamma, R \geq 0, y \in B(x,R)$ *for a fixed independent* $C > 0$ *then there is an* $A_E > 1$ *such that for all* $x \in \Gamma, R > 0$

$$E(x, A_E R) \geq E(x,R). \tag{3.11}$$

Remark 3.1 *It is immdediate from Theorem 3.4 that* (TC) *implies 3.11, which is the anti-doubling property of the mean exit time.*
It is also shown in [20] that

$$E(x,R) \geq cR^2$$

provided (p_0) *and* (VD) *hold furthermore* $E(x,R)$ *for* $R \in \mathbb{N}$ *is strictly monotone and consequently has inverse*

$$e(x,n) = \min\{r \in \mathbb{N} : E(x,r) \geq n\}.$$

It is also worth to recall that the following statements are equivalent
1. There are $C, c > 0, \beta \geq \beta' > 0$ *such that for all* $x \in \Gamma, R \geq S > 0$, $y \in B(x,R)$

$$c\left(\frac{R}{S}\right)^{\beta'} \leq \frac{E(x,R)}{E(y,S)} \leq C\left(\frac{R}{S}\right)^{\beta}, \tag{3.12}$$

2. There are $C, c > 0, \beta \geq \beta' > 0$ *such that for all* $x \in \Gamma, n \geq m > 0$, $y \in B(x, e(x,n))$

$$c\left(\frac{n}{m}\right)^{1/\beta} \leq \frac{e(x,n)}{e(y,m)} \leq C\left(\frac{n}{m}\right)^{1/\beta'}. \tag{3.13}$$

Definition 3.1 *The local sub-Gaussian kernel at* $z \in \Gamma$ *is the following. Let* $k = k_z(n, R) \geq 0$ *the maximal integer for which*

$$\frac{n}{k} \leq qE\left(z, \left\lfloor \frac{R}{k} \right\rfloor\right) \tag{3.14}$$

and $k_z(n, R) = 0$ *if there is no such an integer. The local sub-Gaussian kernel is defined as*

$$k(x, n, R) = \min_{z \in B(x,R)} k_z(n, R).$$

Remark 3.2 *From the definition of* $k_z(n, R)$ *and* (TC) *it follows easily that*

$$k_z(n, R) + 1 \geq c\left(\frac{E(z, R)}{n}\right)^{\frac{1}{\beta - 1}}$$

and

$$k(x, n, R) + 1 \geq c\left(\frac{E(x, R)}{n}\right)^{\frac{1}{\beta - 1}}. \tag{3.15}$$

The equivalence of the isoperimetric inequalities in Theorem 2.1 is based on the next observation.

Proposition 3.5 *Let* $\delta > 0$ *and* $A \subset \Gamma$ *a finite set. The following statements are equivalent.*

$$\overline{E}(A) \leq C\mu(A)^{\delta}, \tag{3.16}$$

$$\lambda^{-1}(A) \leq C\mu(A)^{\delta}, \tag{3.17}$$

$$\rho(D, A)\mu(D) \leq C\mu(A)^{\delta} \text{ for all } D \subset A. \tag{3.18}$$

The proofs are given via a series of lemmas.

Lemma 3.6 *(cf Lemma 4.6 [17]) For all weighted graphs and for all finite stes,* $A \subset B \subset \Gamma$ *the inequality*

$$\lambda(B)\rho(A, B^c)\mu(A) \leq 1, \tag{3.19}$$

holds, particularly

$$\lambda(x, 2R)\rho(x, R, 2R)V(x, R) \leq 1. \tag{3.20}$$

Lemma 3.7 *(cf Proposition 2.2 [18]) If for a finite $A\subset\Gamma$ there are $C,C',\delta>0$ such that*

$$\mu(D)\rho(D,A)\leq C\mu(A)^{\delta}\ \textit{for all } D\subset A \tag{3.21}$$

then,

$$\overline{E}(A)\leq C'\mu(A)^{\delta}.$$

Lemma 3.8 *(c.f Lemma 3.6 [19])For any finite set $A\subset\Gamma$*

$$\lambda^{-1}(A)\leq\overline{E}(A) \tag{3.22}$$

Proof of Proposition 3.5. The implication $(3.16)\Longrightarrow(3.17)$ follows from Lemma 3.8, $(3.17)\Longrightarrow(3.18)$ from Lemma 3.6 and finally $(3.18)\Longrightarrow(3.16)$ by Lemma 3.7. ■

We finish this section showing the connection between the isoperimetric inequalities in Theorem 2.1 and Corollary 2.2.

Proposition 3.9 *The statements $(E),(FK)$ and (ρ) are equivalent as well as $(2.4),(2.5)$ and (2.6).*

Proof. The first statement follows from Proposition 3.5 setting $C=C'\frac{E(x,R)}{V(x,R)^{\delta}}$. The second statement uses Proposition 3.5 and the observation that (2.6) can be written as

$$\rho(D,A)\mu(D)\leq C\rho(x,R,2R)V(x,R)\frac{\mu(A)^{\delta}}{V(x,R)^{\delta}}.$$

■

Proposition 3.10 *Each statement $(E),(FK)$ and (ρ) implies (VD) and (TC).*

Proof. First let us observe that if one does then all of them, since they are equivalent by Proposition 3.9. So we can choose (E). Let $A=B(x,R)$ then $A=B(y,2R)$ we have immediately (VD) and (TC). ■

Proposition 3.10 means that the volume doubling property, (VD) and time comparison principle, (TC) can be set as precondition in Theorem 2.1 as it is done in Corollary 2.2.

Proposition 3.11 *Theorem 2.1 and Corollary 2.2 mutually imply each other.*

Proof. According to Proposition 3.10 we can set (VD) and (TC) as preconditions then using Theorem 3.3 the r.h.s. of each inequality $E(x,R)$ can be replaced with the needed term receiving that $(E)\Longrightarrow(2.4),(FK)\Longrightarrow(2.5)$ and $(\rho)\Longrightarrow(2.6)$. The opposite implications can be seen choosing $R'=\frac{3}{2}R$ and applying $(VD),(TC)$ and (3.5). This clearly gives the statement, if any of the isoperimetric inequalities is equivalent with the diagonal upper estimate then all of them are. ■

4 The upper estimates

In this section we shall show the following theorem, which implies Theorem 2.1 according to Proposition 3.11 and Theorem 3.3.

Theorem 4.1 *If* (Γ,μ) *satisfies* $(p_0),(VC)$ *and* (TC) *then the following statements are equivalent*

$$\lambda^{-1}(A)\le CE(x,R)\left(\frac{\mu(A)}{\mu(B)}\right)^{\delta}\text{ for all }A\subset B(x,2R), \tag{4.23}$$

$$p_n(x,x)\le\frac{C}{V(x,e(x,n))}. \tag{4.24}$$

$$p_n(x,y)\le\frac{C}{V(x,e(x,n))}\exp\left(-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right). \tag{4.25}$$

4.1 Estimate of the Dirichlet heat kernel

Lemma 4.2 *Let* (Γ,μ) *be a weighted graph. Assume that for* $a,C>0$ *fixed constants and for any non-empty finite set* $A\subset\Gamma$

$$\lambda(A)^{-1}\le aC\mu(A)^{\delta}. \tag{4.26}$$

The for any $f(x)$ *non-negative function on* Γ *with finite support*

$$a\|f\|_2^2\left(\frac{\|f\|_2}{\|f\|_1}\right)^{2\delta}\le C\mathcal{E}(f,f).$$

Proof. The proof is simple modification of [10, Lemma 5.2] (see also [9, Lemma 2.2]). ■

Now we have to make a careful detour as it was made in [4] or [10]. The strategy is the following. We consider the weighted graph Γ^* with the same vertex set as Γ with new edges and weights induced by the two-step transition operator $Q=P^2$,

$$\mu^*_{x,y}=\mu(x)P_2(x,y).$$

If Γ^* is decomposed into two disconnected component due to the periodicity of P the applied argument will work irrespective which component is considered. We show that $(p_0),(VD),(TC)$ and (FK) hold on Γ^* if they hold on Γ. We deduce the Dirichlet heat kernel estimate for Q on Γ^* then, we show that it implies the same on Γ. We have to do this detour to ensure

$$\frac{1}{\mu'(x)}Q(x,x)=q(x,x)\ge\alpha>0$$

holds for all $x\in\Gamma^{*}$ which will be needed in the key step to show the diagonal upper estimate in the proof of Lemma 4.6.

Lemma 4.3 *If* $(p_0),(VD),(TC),(FK)$ *holds on* Γ *then, the same is true on* Γ^{*}.

Proof. The statement is evident for (p_0) and (VD). Here it is worth to mention that $\mu^{*}(x)=\mu(x)$ and from (3.7) we know that $\mu(x)\simeq\mu(y)$ if $x\sim y$. Let us observe that

$$\begin{aligned} B(x,2R) &\subset \overline{B}^{*}(x,R) && (4.27)\\ B^{*}(x,R) &\subset B(x,2R) && (4.28)\end{aligned}$$

and

$$\begin{aligned} V^{*}(y,2R) &\leq V(y,4R)\leq C^{2}V(x,R) && (4.29)\\ &\leq C^{2}\mu\left(\overline{B}^{*}(x,R/2)\right)\leq C^{2}V^{*}(x,R).\end{aligned}$$

So, not only (VD) can be shown but the volumes of the above balls are comparable.

The next is to show (TC).

$$\begin{aligned} E^{*}(y,2R) &= \sum_{z\in B^{*}(y,2R)}\sum_{k=0}^{\infty}Q_{k}^{B^{*}(y,2R)}(y,z)\\ &\leq \sum_{z\in B(y,4R)}\sum_{k=0}^{\infty}P_{2k}^{B(y,4R)}(y,z)\\ &\leq \sum_{z\in B(y,4R)}\sum_{k=0}^{\infty}P_{2k}^{B(y,4R)}(y,z)+P_{2k+1}^{B(y,4R)}(y,z)\\ &= E(y,4R)\leq CE(x,R/2)\\ &= \sum_{z\in B(x,R/2)}\sum_{k=0}^{\infty}P_{k}^{B(x,R/2)}(x,z)\\ &= \sum_{z\in B(x,R/2)}\sum_{k=0}^{\infty}P_{2k}^{B(x,R/2)}(x,z)+P_{2k+1}^{B(x,R/2)}(x,z).\end{aligned}$$

Now we use a trivial estimate.

$$\begin{aligned} P_{2k+1}^{B(x,R)}(x,z) &= \sum_{w\sim z}P_{2k}^{B(x,R)}(x,w)\,P^{B(x,R)}(w,z)\\ &\leq \sum_{w\sim z}P_{2k}^{B(x,R)}(x,w).\end{aligned}$$

Summing up for all z and recalling (3.8) which states that for a fixed $w \in \Gamma$, $|\{w \sim z\}| \leq \frac{1}{p_0}$, we receive that

$$\begin{aligned}\sum_{z\in B(x,R/2)} P_{2k+1}^{B(x,R/2)}(x,z) &\leq \sum_{z\in B(x,R/2)}\sum_{w\sim z} P_{2k}^{B(x,R/2)}(x,w) \\ &\leq C\sum_{w\in \overline{B}(x,R/2)} P_{2k}^{B(x,R)}(x,w).\end{aligned}$$

As a result we obtain that

$$\begin{aligned}E^*(y,2R) &\leq C\sum_{z\in\overline{B}(x,R/2)}\sum_{k=0}^{\infty} P_{2k}^{\overline{B}(x,R/2)}(x,z) \\ &\leq CE^*(x,R/2+1) \leq CE^*(x,R).\end{aligned}$$

This shows that (TC) holds on Γ^*. We have also proved that

$$cE^*(x,R) \leq E(x,R) \leq CE^*(x,R). \tag{4.30}$$

It is left to show that from

$$\lambda(A)^{-1} \leq CE(x,R)\left(\frac{\mu(A)}{V(x,R)}\right)^{\delta} \tag{4.31}$$

it follows that

$$\lambda^*(A)^{-1} \leq CE^*(x,R)\left(\frac{\mu^*(A)}{V^*(x,R)}\right)^{\delta} \tag{4.32}$$

holds as well. The inequality

$$\lambda^*(A) \geq \lambda\left(\overline{A}\right) \tag{4.33}$$

was given in [4, Lemma 4.3]. Collecting the inequalities we get the statement.

$$\begin{aligned}\lambda^*(A)^{-1} &\leq \lambda\left(\overline{A}\right)^{-1} \leq CE(x,R+1)\left(\frac{\mu\left(\overline{A}\right)}{V(x,R+1)}\right)^{\delta} \\ &\leq CE^*(x,R)\left(\frac{\mu^*(A)}{V^*(x,R)}\right)^{\delta}\end{aligned}$$

■

Lemma 4.4 *For all random walks on weighted graphs, $x,y \in A \subseteq \Gamma$, $n,m \geq 0$*

$$p_{n+m}^{A}(x,y) \leq \sqrt{p_{2n}^{A}(x,x)\,p_{2m}^{A}(y,y)}. \tag{4.34}$$

Proof. The proof is standard, hence omitted. ■

To complete the scheme of the proof we need to return from Γ^* to Γ. This is given in the following lemma.

Lemma 4.5 *Assume that* (Γ,μ) *satisfy* $(p_0)\,(VD)$ *and* (TC). *In addition if* (DUE) *holds on* (Γ^*,μ) *then, it holds* (Γ,μ).

Proof. The condition states that

$$q_n\left(x,x\right)\leq\frac{C}{V^*\left(x,e^*\left(x,n\right)\right)}.$$

Then from the definition of q, (4.29) and (4.30) it follows that

$$p_{2n}\left(x,x\right)\leq\frac{C}{V\left(x,e\left(x,2n\right)\right)}.$$

Finally for odd times the statement follows by a standard argument. From the spectral decomposition of $P_n^{B(x,R)}$ for finite balls one has that

$$P_{2n}^{B(x,R)}\left(x,x\right)\geq P_{2n+1}^{B(x,R)}\left(x,x\right)$$

and consequently

$$\begin{aligned}p_{2n}\left(x,x\right)&=\lim_{R\to\infty}p_{2n}^{B(x,R)}\left(x,x\right)\\&\geq\lim_{R\to\infty}p_{2n+1}^{B(x,R)}\left(x,x\right)=p_{2n+1}\left(x,x\right),\end{aligned}$$

which gives the statement using $(VD)\,,(TC)$ and (3.13). ■

Lemma 4.6 *If* (p_0) *is true on* (Γ,μ) *and* (FK) :

$$\lambda(A)^{-1}\leq Ca\mu\left(A\right)^{\delta}\tag{4.35}$$

holds for

$$a=\frac{E\left(x,R\right)}{V\left(x,R\right)^{\delta}}\simeq\frac{E^*\left(x,R\right)}{V^*\left(x,R\right)^{\delta}}$$

and all $A\subset B^*\left(x,R\right)$ *on* (Γ^*,μ) *then, for all* $x,y\in\Gamma,n>0$

$$q_n^{B^*(x,R)}\left(y,y\right)\leq C\left(\frac{a}{n}\right)^{1/\delta}.$$

Proof. The proof is slight modification of the steps proving $(a) \Longrightarrow (b)$ in Proposition 5.1 of [10] so we omitt it. This final statement can be reformulated for $y \in \Gamma$ as follows

$$q_{2n}^{B^*}(y,y) \leq \frac{C}{V(x,R)} \left(\frac{E(x,R)}{n}\right)^{1/\delta}.$$

■

Now we consider the following path decompositions.

Lemma 4.7 *Let $p_n(x,y)$ the heat kernel of the random walk on an arbitrary weighted graph (Γ,μ). Let $A \subset \Gamma, x,y \in A, n > 0$ then*

$$p_n(x,y) \leq p_n^A(x,y) + P_x(T_A < n) \max_{\substack{z\in\partial A \\ 0\leq k<n}} p_k(z,y), \tag{4.36}$$

$$\begin{aligned} p_n(x,y) &\leq p_n^A(x,y) + P_x(T_A < n/2) \max_{\substack{z\in\partial A \\ n/2\leq k<n}} p_k(z,y) & (4.37)\\ &+ P_y(T_A < n/2) \max_{\substack{z\in\partial A \\ n/2\leq k<n}} p_k(z,x). & (4.38)\end{aligned}$$

Proof. Both inequality follows as in [9, Lemma 2.5] from the first exit decomposition starting from x or from x and y as well. ■

4.2 Proof of the upper estimates

Proof of Theorem 4.1. First we show the implication $(FK) \Longrightarrow (DUE)$ on Γ^* assuming $(p_0), (VD)$ and (TC). We follow the main lines of [9]. Let we choose r so that $Ln = E(x,r)$ for a large $L > 0$. From (4.37) we have that for $B = B^*(x,r)$

$$q_n(x,x) \leq q_n^B(x,x) + 2Q_x(T_B < n/A) \max_{\substack{z\in\partial B \\ n/A\leq k<n}} q_k(z,x). \tag{4.39}$$

From (4.34) one gets that for all $n/A \leq k < n$

$$q_k(z,x) \leq \sqrt{q_k(z,z)\, q_k(x,x)} \leq \max_{v\in\overline{B}} q_k(v,v) \leq C_1 \max_{v\in\overline{B}} q_{\lfloor n/A\rfloor}(v,v).$$

This results in (4.39) that for some $x_1 \in \overline{B}$

$$q_n(x,x) \leq q_n^B(x,x) + 2Q_x(T_B < n/A)\, C_1 q_{\lfloor n/A\rfloor}(x_1,x_1). \tag{4.40}$$

We continue this procedure. In the i-s step we have

$$q_{n_{i-1}}(x_i,x_i) \le q_{n_i}^{B_i}(x_i,x_i) + 2Q_{x_i}(T_{B_i} < n_{i+1})\, C_1 q_{n_i}(x_{i+1},x_{i+1}), \qquad (4.41)$$

where $n_i = \lfloor n/A^i \rfloor, r_i = e(x_i, Ln_i), B_i = B(x_i,r_i), x_{i+1} \in \overline{B}_i$. Let $m = \lfloor \log_A n \rfloor$ and we stop the iteration at m. This means that $1 \le \left\lfloor \frac{n}{A^m} \right\rfloor = n_m < A$. Now we choose A. By the definition of n_i and from (TC) it follows that

$$A = \frac{Ln_i}{Ln_{i+1}} = \frac{E(x_i,r_i)}{E(x_{i+1},r_{i+1})} \le \frac{E(x_i,2r_i)}{E(x_{i+1},r_{i+1})} \le C\left(\frac{2r_i}{r_{i+1}}\right)^{\beta},$$

which results with $\sigma = 2\left(\frac{C}{A}\right)^{\frac{1}{\beta}} < 1/2$ that

$$r_{i+1} \le \sigma r_i \qquad (4.42)$$

if $A > 4^{\beta}C$. From the choice of the constants it follows that

$$d(x,x_m) \le r + r_1 + ... + r_m \le r\sum_{i=0}^{m}\sigma^i < r\frac{1}{1-\sigma} \le 2r. \qquad (4.43)$$

From Lemma 4.6 the first term can be estimated as follows

$$\begin{aligned} q_{n_i}^{B_i}(x_i,x_i) &\le \frac{C}{V(x_i,r_i)}\left(\frac{E(x,r_i)}{n_i}\right)^{1/\delta} \\ &= \frac{C}{V(x_i,r_i)}L^{1/\delta} = \frac{CL^{1/\delta}}{V(x,2r)}\frac{V(x,2r)}{V(x_i,r_i)} \\ &\le \frac{CL^{1/\delta}}{V(x,r)}\left(\frac{2}{\sigma}\right)^{\alpha i}, \end{aligned}$$

where in the last step (4.43) and (VD) have been used. Let us observe that by definition of $k = k(x_i, n_{i+1}, r_i)$

$$\frac{n_{i+1}}{k+1} > \min_{y\in B_i} E\left(y, \frac{r_i}{k+1}\right)$$

and by (TC)

$$\begin{aligned} Ln_i &= E(x_i,r_i) \le CE(y,r_i) \le C(k+1)^{\beta} E\left(y,\frac{r_i}{k+1}\right) \\ &\le C(k+1)^{\beta-1} n_i, \end{aligned}$$

which results that $k > (L/C)^{\frac{1}{\beta-1}} - 1$ and

$$Q_{x_i}(T_B < n_{i+1}) \le C\exp\left[-ck(x_i,n_{i+1},r_i)\right] \le C\exp\left[-c\left(\frac{L}{C}\right)^{\frac{1}{\beta-1}}\right].$$

This means that

$$Q_{x_i}\left(T_B < n_{i+1}\right) \leq \frac{\varepsilon}{2}$$

if L is chosen to be enough large. Inserting this into (4.41) one gets

$$q_{n_{i-1}}\left(x_i, x_i\right) \leq \frac{CL^{1/\delta}}{V\left(x,r\right)}\left(\frac{2}{\sigma}\right)^{\alpha i} + \varepsilon C_1 q_{n_i}\left(x_{i+1}, x_{i+1}\right). \tag{4.44}$$

Summing up the iteration results that

$$q_n\left(x,x\right) \leq \frac{CL^{1/\delta}}{V\left(x,r\right)}\sum_{i=1}^{m}\left(\left(\frac{2}{\sigma}\right)^{\alpha}\varepsilon\right)^{i} + \left(\varepsilon C_1\right)^m q_m\left(x_m, x_m\right). \tag{4.45}$$

Choosing L enough large $\varepsilon < \min\left\{\left(\frac{\sigma}{2}\right)^{\alpha}, \frac{1}{C_1}\right\}$ can be ensured. This results that the sum in the first term is bounded by $1/\left(1-\varepsilon\left(\frac{2}{\sigma}\right)^{\alpha}\right) < C$. The second term can be treated as follows.

$$q_m\left(x_m, x_m\right) = \frac{1}{\mu\left(x_m\right)}Q_m\left(x_m, x_m\right) \leq \frac{1}{\mu\left(x_m\right)}.$$

From (4.42) we have that

$$\begin{aligned}\frac{1}{\mu\left(x_m\right)} &= \frac{1}{V\left(x,r\right)}\frac{V\left(x,2r\right)}{\mu\left(x_m\right)}\\ &\leq \frac{1}{V\left(x,r\right)}\left(2r\right)^{\alpha}.\end{aligned}$$

This means that we are ready if

$$\left(2r\right)^{\alpha}\varepsilon^m < C'.$$

Let us remark that $E\left(z,r\right) \geq r$. which implies that $e\left(x,n\right) \leq n$. From the definition of m and $E\left(x,r\right) = Ln$,

$$\left(2r\right)^{\alpha}\varepsilon^m \leq \left(2r\right)^{\alpha}\varepsilon^{\log_A n} \leq \left[2E\left(x,r\right)\right]^{\alpha} n^{\log_A \varepsilon} = \left(2L\right)^{\alpha} n^{\alpha + \log_A \varepsilon} \leq C$$

if $\varepsilon < A^{-\alpha}$, L is enough large. Finally from (4.45) we receive that

$$\begin{aligned} q_n\left(x,x\right) &\leq \frac{CL^{1/\delta}}{V\left(x,r\right)}\sum_{i=1}^{m}\left(2^{\alpha}\varepsilon\right)^{i} + \left(\varepsilon C_1\right)^m q_{n_m}\left(x_m, x_m\right) && (4.46)\\ &\leq \frac{C}{V\left(x,r\right)} = \frac{C}{V\left(x, e\left(x, Ln\right)\right)} \leq \frac{C}{V\left(x, e\left(x,n\right)\right)}, && (4.47)\end{aligned}$$

if $\varepsilon < \min\left\{\left(\frac{\sigma}{2}\right)^{\alpha}, \frac{1}{C_1}, A^{-\alpha}\right\}$, absorbing all the constants into C. This means that (DUE) holds on Γ^* and by Lemma (4.5) (DUE) holds on Γ as well. It was shown in [21] that under the assumption (p_0)

$$(VD)+(TC)+(DUE)\Longrightarrow(UE).$$

The revers implication $(UE)\Longrightarrow(DUE)$ is evident. The implication $(DUE)\Longrightarrow(FK)$ can be seen as it was given in [4] without any essential change, hence the proof of Theorem 4.1 and2.1 is complete. . ■

4.3 A Davies-Gaffney type upper estimate

We provide here a different proof of the upper estimate which might be interesting on its own. The proof has two ingredients. The first one is the generalization of the Davies-Gaffney inequality. First we need a theorem from [19].

Theorem 4.8 *If (p_0) and $\left(\overline{E}\right)$ hold then there are $c, C>0$ such that for all $x\in\Gamma, n, R>0$*

$$\mathbb{P}_x(T_{x,R}<n)\leq C\exp\left[-ck\left(x,n,R\right)\right]. \tag{4.48}$$

Proof. See Theorem 5.1 [19]. ■

Notation 3 *Denote*

$$k\left(n,A,B\right)=\min_{z\in A}k\left(z,n,d\right), \tag{4.49}$$

where $d=d\left(A,B\right)$ and

$$\kappa\left(n,A,B\right)=\max\left\{k\left(n,A,B\right),k\left(n,B,A\right)\right\}. \tag{4.50}$$

Theorem 4.9 *If $\left(\overline{E}\right)$ holds for a reversible Markov chain then there is a constant $c>0$ such that for all $A,B\subset V$, the Davies-Gaffney type inequality (DG)*

$$\sum_{x\in A,y\in B}p_n(x,y)\mu(x)\mu\left(y\right)\leq\left[\mu\left(A\right)\mu\left(B\right)\right]^{1/2}\exp\left(-c\kappa\left(n,A,B\right)\right) \tag{DG}$$

holds.

Proof. Using the Chebisev inequality one gets

$$
\begin{aligned}
& \sum_{x\in A, y\in B} P_n(x,y)\mu(x) && (4.51)\\
= & \sum_{x\in\Gamma}\mu\left(x\right)^{1/2} I_A\left(x\right)\left[\mu^{1/2}(x)I_A\left(x\right)\sum_{y\in B}P_n(x,y)I_B\left(y\right)\right] && (4.52)\\
\leq & \left(\mu\left(A\right)\right)^{1/2}\left\{\sum_{x\in\Gamma}\mu\left(x\right)I_A\left(x\right)\left[\sum_{y\in\Gamma}P_n\left(x,y\right)I_B\left(y\right)\right]^2\right\}^{1/2}.
\end{aligned}
$$

Let us deal with the second term denoting $r = d\left(A,B\right)$

$$
\begin{aligned}
& \sum_{x\in\Gamma}\mu\left(x\right)I_A\left(x\right)\left[\sum_{y\in\Gamma}P_n\left(x,y\right)I_B\left(y\right)\right]^2 && (4.53)\\
= & \sum_{x\in\Gamma}\mu\left(x\right)I_A\left(x\right)\sum_{y\in\Gamma}P_n\left(x,y\right)I_B\left(y\right)\sum_{z\in\Gamma}P_n\left(x,z\right)I_B\left(z\right)\\
= & \sum_{x\in\Gamma}\sum_{y\in\Gamma}\sum_{z\in\Gamma}P_n\left(x,z\right)I_B\left(z\right)\mu\left(x\right)I_A\left(x\right)P_n\left(x,y\right)I_B\left(y\right) && (4.54)\\
= & \sum_{z\in\Gamma}\sum_{y\in\Gamma}\sum_{x\in\Gamma}P_n\left(z,x\right)I_B\left(z\right)\mu\left(z\right)I_A\left(x\right)P_n\left(x,y\right)I_B\left(y\right) && (4.55)\\
\leq & \sum_{z\in\Gamma}\sum_{x\in\Gamma}P_n\left(z,x\right)I_B\left(z\right)\mu\left(z\right)I_A\left(x\right)\sum_{y\in\Gamma}P_n\left(x,y\right)I_B\left(y\right)\\
\leq & \sum_{z\in\Gamma}\sum_{x\in\Gamma}P_n\left(z,x\right)I_B\left(z\right)\mu\left(z\right)I_A\left(x\right) && (4.56)\\
\leq & \sum_{z\in\Gamma}P_n\left(z,A\right)I_B\left(z\right)\mu\left(z\right)\leq\sum_{z\in\Gamma}P_z(T_{z,r}<n)I_B\left(z\right)\mu\left(z\right)\\
\leq & \max_{z\in B}\exp\left[-ck\left(z,n,r\right)\right]\mu\left(B\right). && (4.57)
\end{aligned}
$$

The combination of (4.51) and (4.53) gives the second term in the definition of κ and by symmetry one can obtain the first one. ■

4.4 The parabolic mean value inequality

In order to show the off-diagonal upper estimate we need that the so called parabolic mean value (PMV) inequality follows from the diagonal upper estimate. Working under the conditions $(p_0), (VD)$ and (TC) we will show the following implications

$$(DUE) \Longrightarrow (PMV)$$

and

$$(PMV)+(DG)\Longrightarrow(UE).$$

In doing so we introduce (PMV).

Definition 4.1 *A weighted graph satisfies the parabolic mean value inequality* (**PMV**) *if for fixed constants* $0<c_1<c_2$ *there is a* $C>1$ *such that for arbitrary* $x\in\Gamma$ *and* $R>0$, *using the notations* $E=E(x,R),B=B(x,R),n=c_2E,\Psi=[0,n]\times B$ *for any non-negative Dirichlet solution of the heat equation*

$$P^Bu_i=u_{i+1}$$

on Ψ, *the inequality*

$$u_n(x)\leq\frac{C}{E(x,R)V(x,R)}\sum_{i=c_1E}^{c_2E}\sum_{y\in B(x,R)}u_i(y)\mu(y) \tag{4.58}$$

holds.

Theorem 4.10 *If* (Γ,μ) *satisfies* $(p_0),(VD)$ *and* (TC) *then,*

$$(DUE)\Longrightarrow(PMV)$$

Proof. For the proof see [21]. ■

Remark 4.1 *Let us observe that if for non-negative Dirichlet (sub-)solutions the parabolic mean value inequality holds then it holds on non-negative (sub-)solutions as well. This can be seen by the decomposition of an* $u\geq0$ *solution on* $B(x,2R)$ *on the smaller ball* $B(x,R)$ *into nonnegative combination of non-negative Dirichlet solutions in* $B(x,2R)$. *(c.f. [7]).*

4.5 The local upper estimates

Proposition 4.11 *Assume that* (Γ,μ) *satisfies* (p_0), (PMV) *and* (TC). *Let* $x,y\in\Gamma$ *then there are* $c,C>0,\beta>1$ *such that for all* $x,y\in\Gamma,n>0$

$$p_n(x,y)\leq\frac{C}{\sqrt{V(x,e(x,n))V(y,e(y,n))}}\exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{4.1}$$

Proof. The proof combines the repeated use of the parabolic mean value inequality and the the Davies-Gaffney inequality. Following the idea of [15]. Denote $R = e(x,n), S = e(y,n)$ and assume that $d \geq \frac{2}{3}(R+S)$ which ensures that $r = d - R - S \geq \frac{1}{3}d$. From (PMV) it follows that

$$p_n(x,y) \leq \frac{C}{V(x,R)E(x,R)} \sum_{c_1 E(x,R)}^{c_2 E(x,R)} \sum_{z\in B(x,R)} p_k(z,y)\mu(z)$$

and using (PMV) for $p_k(z,y)$ on gets

$$p_n(x,y) \leq \frac{C}{V(x,R)V(y,S)n^2} \sum_{i=c_1 n}^{c_2 n} \sum_{z\in V(x,R)} \sum_{j=c_1 n+i}^{c_2 n+i} \sum_{w\in B(y,S)} p_j(z,w)\mu(z)\mu(w) \tag{4.2}$$

Now by (DG) and (3.15) and denoting $A = B(x,R), B = B(y,S)$ we obtain

$$p_n(x,y) \leq \frac{C\sqrt{V(x,R)V(y,S)}}{V(x,R)V(y,S)n^2} \sum_{i=c_1 n}^{c_2 n} \sum_{j=c_1 n+i}^{c_2 n+i} e^{-c\kappa(n,A,B)}. \tag{4.3}$$

Using (TC) and $R < \frac{3}{2}d$ one can see that

$$\max_{z\in V(x,R)} \exp -c\left(\frac{E(z,d/3)}{n}\right)^{\frac{1}{\beta-1}} \leq \exp -c\left(\frac{E(x,d)}{n}\right)^{\frac{1}{\beta-1}}$$

and similarly

$$\begin{aligned} \max_{w\in V(y,R)} \exp -c\left(\frac{E(w,d/3)}{n}\right)^{\frac{1}{\beta-1}} &\leq \exp -c\left(\frac{E(y,d)}{n}\right)^{\frac{1}{\beta-1}} \\ &\leq \exp -c\left(\frac{E(x,d)}{n}\right)^{\frac{1}{\beta-1}} \end{aligned}$$

which results that

$$p_n(x,y) \leq \frac{C}{\sqrt{V(x,R)V(y,S)}} \exp\left[-c\left(\frac{E(x,d)}{n}\right)^{\frac{1}{\beta-1}}\right].$$

It is left to treat the case $d(x,y) < \frac{2}{3}(R+S)$. In this case $\kappa(n, B(x,R), B(y,S)) = 0$ in (4.3). On the other hand if $d(x,y) < R$

$$E(x,d) \leq E(x, e(x,n)) = n$$

which results that $1 \leq C \exp\left[-c\left(\frac{E(x,d)}{n}\right)^{\frac{1}{\beta-1}}\right]$ for a fixed $C>0$ and similarly if $d(x,y)<S$, $1 \leq C \exp\left[-c\left(\frac{E(y,d)}{n}\right)^{\frac{1}{\beta-1}}\right] \leq C \exp\left[-c\left(\frac{E(x,d)}{n}\right)^{\frac{1}{\beta-1}}\right]$ which gives the statement. ■

The next lemma is from [21], which leads to the upper estimate.

Lemma 4.12 *If* $(p_0), (VD)$ *and* (TC) *hold then for all* $\varepsilon > 0$ *there are* $C_\varepsilon, C > 0$ *such that for all* $n > 0, x, y \in \Gamma, d = d(x,y)$

$$\sqrt{\frac{V(x,e(x,n))}{V(y,e(y,n))}} \leq C_\varepsilon \exp \varepsilon C \left(\frac{E(x,d)}{n}\right)^{\frac{1}{(\beta-1)}}.$$

Theorem 4.13 *Assume that* (Γ,μ) *satisfies* (p_0), $(VD), (TC)$ *and* (DUE). *Let* $x, y \in \Gamma$ *then,* (UE) *holds:*

$$p_n(x,y) \leq \frac{C}{V(x,e(x,n))} \exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right]. \tag{4.4}$$

Proof. From Theorem 4.10 we have that

$$(DUE) \Longrightarrow (PMV).$$

Now we can use Proposition 4.11 which statates that from (PMV) and (TC) it follows that

$$p_n(x,y) \leq \frac{C}{\sqrt{V(x,e(x,n))\,V(y,e(y,n))}} \exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right].$$

Let us use Lemma 4.12,

$$\begin{aligned} p_n(x,y) &\leq \frac{C}{V(x,e(x,n))}\sqrt{\frac{V(x,e(x,n))}{V(y,e(y,n))}} \exp\left[-c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right] \\ &\leq \frac{CC_\varepsilon}{V(x,e(x,n))} \exp\left[\varepsilon C\left(\frac{E(x,r)}{n}\right)^{\frac{1}{(\beta-1)}} - c\left(\frac{E(x,d(x,y))}{n}\right)^{\frac{1}{\beta-1}}\right] \end{aligned}$$

and choosing ε small enough we get the statement. ■

5 Example

In this section we recall from [21] an example for a graph which is not covered by any of the previous results of on- and off-diagonal upper estimates but satisfies the conditions of Theorem 2.1.

Let G_i is the subgraph of the Vicsek tree (see Figure 1.) (cf. [11]) which contains the root z_0 and has diameter $D_i = 23^i$. Let us denote by z_i the vertices on the infinite path with $d(z_0, z_i) = D_i$. Denote $G_i' = (G_i \backslash G_{i-1}) \cup \{z_{i-1}\}$ for $i > 0$, the annulus defined by G-s.

The new graph is defined by stretching the Vicsek tree as follows. Consider the subgraphs G_i' and replace all the edges of them by a path of length $i+1$. Denote the new subgraph by A_i, the new blocks by $\Gamma_i = \cup_{j=0}^{i} A_i$, the new graph is $\Gamma = \cup_{j=0}^{\infty} A_j$. We denote by z_i the cut point between A_i and A_{i-1} again. For $x \neq y, x \sim y$ let $\mu_{x,y} = 1$.

One can see that neither the volume nor the mean exit time grows polynomially on Γ, on the other hand Γ is a tree and the resistance grows asymptotically linearly on it.

It was shown in [21] that the tree Γ satisfies $(p_0), (VD), (TC)$ furthermore the mean value inequality (for all the definitions and details see [21]). The main result there states that under these conditions the diagonal upper estimate holds. Since Γ satisfies $(p_0), (VD), (TC)$ and (DUE) we are in the scope of Theorem 2.1 and all the isoperimetric inequalities hold.

References

[1] Barlow, M.T., Which values of the volume growth and escape time exponent are possible for a graph? Revista Mat. Iber, 20 (2004), no. 1, 1–31

[2] Carron, G., Inégalités isopérimétriques de Faber-Krahn et conséequences, Actes de la table ronde de géométrie différentielle (Luminy, 1992), Collection SMF Séminaires et Congrées} 1, (1996) 205–232

[3] Chavel, I., Isoperimetric Inequalities : Differential Geometric and Analytic Perspectives (Cambridge Tracts in Mathematics, No 145) 2001

[4] Coulhon, T., Grigor'yan, A. Random walks on graphs with regular volume growth, Geometry and Functional Analysis, 8, (1998) 656-701

[5] Grigor'yan, A., Coulhon, T. Pointwise estimates for transition probabilities of random walks on infinite graphs, to appear in Proceedings of the Conference "Fractals in Graz" (2002)

[6] Davies, E.B., Heat kernel bounds, conservation of probability and the Feller property, J. Anal. Math. 58 (1992), 99-119

[7] Delmotte, T., Parabolic Harnack inequality and estimates of Markov chains on graphs.Revista Mat. Iber., 1,(1999), 181–232.

[8] Grigor'yan, A., Heat kernel upper bounds on a complete non-compact manifold, Revista Math. Iber., 10, (1994) no.2, 395-452.

[9] Grigor'yan, A., Heat kernel upper bounds on fractal spaces preprint

[10] Grigor'yan, A., Telcs, A., Sub-Gaussian estimates of heat kernels on infinite graphs, Duke Math. J., 109, 3, (2001), 452-510

[11] Grigor'yan, A., Telcs, A., Harnack inequalities and sub-Gaussian estimates for random walks, Math. Annal. 324, (2002) 521-556

[12] Grigor'yan, A., Telcs, A., Heat kernel estimates on measure metric spaces (in preparation)

[13] Kigami, J., Local Nash inequality and inhomogeneity of heat kernels, preprint

[14] Kigami, J., Analysis on fractals, Cambridge Tracts in Mathematics, 143 , Cambridge Univ. Press, 2001

[15] Li, P., Wang, J., Mean value inequalities, Indiana Univ. Math., J., 48, 4, (1999), 1257-1283

[16] Telcs, A., Random Walks on Graphs, Electric Networks and Fractals, Prob. Theo. and Rel. Fields, 82, (1989) 435-449

[17] Telcs, A., Local Sub-Gaussian Estimates on Graphs: The Strongly Recurrent Case, Electronic Journal of Probability, Vol. 6, (2001), Paper no. 22, 1-33

[18] Telcs, A., Isoperimetric inequalities for Random Walks, Potential Analysis, 19, 2003, 237-249

[19] Telcs, A., Volume and time doubling of graphs and random walk, the strongly recurrent case, Communication on Pure and Applied Mathematics, LIV, (2001), 975-1018

[20] Telcs, A., Some notes on the Einstein Relation, preprint

[21] Telcs, A., Random walks on graphs with volume and time doubling, submitted

[22] Varopoulos, R., Saloff-Coste, L., Coulhon, Th., Analysis and geometry on Groups, Cambridge University Press, 1993