Branching and Tree Indexed Random Walks on Fractals

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Abstract

This paper deals with recurrence of branching random walks on polynomially growing graphs. Amongst other things, we demonstrate strong recurrence of tree indexed random walks determined by the resistance properties of spherically symmetric graphs. Several branching walk models are considered to show how the branching mechanism influence the recurrence behaviour.

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1 Introduction

Branching Random Walks (BRWs) are a special type of branching diffusion processes (see [H1] as basic reference). Recent papers have investigated BRWs on different structures like trees, groups and random environments (basically randomly labelled trees). This paper extends this scope to fractals. Fractals are objects of intensive research activity in mathematics and physics. (See [RT],[B1] for basic references.) Our interest is given to BRWs with polynomial growth.

We define BRWs at first in a general way. Let $\mathcal{G}$ denote a set of infinite, locally bounded connected graphs $\{G = (V, E)\}$ without loops (edges connecting a vertex $x \in V$ to itself). Let $deg(x)$ denote the degree of the vertex x and $D > 0$ the uniform upper bound on vertex degrees in G. Let $d(x, y)$ denote the usual graph distance.

The simple nearest neighbour random walk (RW) on $G \in \mathcal{G}$ is a discrete time Markov chain X_n on V whose transition probability matrix has nonzero entries only for $(x, y) \in E$ given by

$$\mathbf{P}(x, y) = \frac{1}{deg(x)}.$$

The n-step transition probability will be denoted by $p_n(x, y)$. The BRW on G is a set of particles moving on V independently by the rule p. These particles give birth to others and die under the control of a birth-death mechanism. The BRW starts from $x \in V$ with a single particle, and a particle reaching $y \in V$ at time n gives birth to k other particles with the probability

$$f(k, \mathcal{F}_n)$$

where $\mathcal{F}_n$ is the sigma algebra of all possible outcomes until time n, in other words, the full past.

In this paper we apply two restrictions always. We consider only the so-called local birth models, where all the offspring start their own life at y. Also, we consider pure birth processes only, but some of our results easily generalise to birth-death processes as well.

We shall use the notions of recurrence and strong recurrence in the sense used in [BP2]; that is, a Markov chain is recurrent if infinitely many returns to a given site occur with positive probability. The recurrence is strong if this probability is one.

Schinazi [S] analysed BRWs and branching Markov chains in the case of independent birth and death systems with constant birth and death rates. He found that the recurrence has a critical value with respect to the reproduction rate, but his model deals with exponentially growing families only. Benjamini and Peres [BP1] provided a very useful tool to analyse BRWs: tree-indexing. They also found criteria for recurrence in terms of the growth of the index tree. They consider the d-dimensional integer lattice RW and the general case of RWs on groups with polynomial growth. Their result shows that some BRWs can be recurrent only if the associated BP (Branching Process) is polynomially growing. In [BP1, Theorem 1.3] one can find more results on tree indexed Markov chains, and for this paper we use their definition. This is applicable for Tree Indexed Random Walks (TIRWs) on graphs as well. One of our results generalises [BP2] and Proposition 6.1 of [BP1], and shows the role of the resistance properties of the graph.

We list some important types of BRWs, classified by the value of

$$f(k, \mathcal{G}_n),$$

without attempting completeness, where g denotes an arbitrary function:

(a) $\mathcal{G}_n$ is the total past of all the particles;

(b) $\mathcal{G}_n$ is the individual history of the traced particle reaching y;

(c)
$$f(k, \mathcal{G}_n) = g(k, \bigcup_{x \in V} \mathcal{G}_{x,n})$$
where $\mathcal{G}_{x,n}$ stands for the histories of the particles reaching x at time n;

(d)
$$f(k, \mathcal{G}_n) = g(k, x, a)$$
where a is the age of the traced particle (subcase of (b) and (c));

(e) for all $x \in V$
$$f(k, \mathcal{G}_n) = g(k, x);$$
(We call this the pure site dependent model.)

(f)

$$f(k, \mathcal{G}_n) = g(k, a).$$

(We call this the pure age-dependent model.)

In the literature of BPs several models are tackled. An interesting model of case (a) is the population size dependent one; c.f. [K].

TIRWs are a special type of BRWs where the BP is independent of the location and history of the individuals. A TIRW is defined by the prescibed branching tree of the family (and the RW on the graph). The vertex set of each level Γ_n of the rooted tree Γ corresponds to the set of particles at time n which make independent steps by the law of the RW on the graph.

In the rest of this section we give basic concepts, definitions needed later and the main results of the paper. Section 2 contains examples of particular processes and Section 3 gives the proofs of the results.

We use the notation $S_{x,N} = \{y \in V : d(x,y) = N\}$, $S_N = S_{0,N}$, $B_{x,N} = \{y \in V : d(x,y) \leq N\}$, $B_N = B_{0,N}$. To avoid the technical problems we shall say that $d = d(a)$ is the *exponent* of a monotonic sequence $\{a_n\}$ if for an appropriate constant A,

$$\lim_{n\to\infty} \frac{\log(a_n - A)}{\log n} = d(a).$$

We shall refer to this situation by the notation

$$a_n \sim A + Cn^d.$$

We say that $G \in \mathcal{G}$ is *polynomially growing* if the volume of the ball $B_{x,N}$ is $O(N^d)$ for some fixed d. We call the infimum of such d the fractal dimension of G. The set of all graphs with this fractal dimension will be denoted by $\mathcal{P}_d$, and the union of this over d by $\mathcal{P}$; that is, $\mathcal{P}$ is the set of all polynomially growing graphs in the strict sense defined above. For a BRW we use the term *exponentially growing* if for some $\epsilon > 0$ the expected size of the population is greater than $(1+\epsilon)^n$ (all n sufficiently large). If this growth rate is with probability 1 rather than in expectation, we say it occurs almost surely. Our first two results give a quite superficial investigation of the relationship between recurrence and growth rate of the BRW, and are followed by a deeper examination of a TIRW.

Theorem 1 *For any pure site-dependent BRW on $G \in \mathcal{G}$ such that at least one site has a non-zero probability of births, if the BRW is recurrent then it is exponentially growing.*

The converse of Theorem 1 is clearly false, but one can ask what extra conditions on the graph or BRW are enough to ensure that a recurrent walk is exponentially growing. Before giving an answer, let us recall Schinazi's result [S], which can be interpreted in the following way. Consider a homogeneous birth and death parameter BP over a countable set Markov chain. Let Z_n denote the (random) set of the family at time n. In [S] it was supposed that the n-step transition probability satisfies

$$p_n(x,x) \sim c_1 e^{-\gamma n},$$

and concluded that the branching Markov chain is recurrent iff

$$\mathbf{E}(|Z_n|) \sim c_2 e^{\alpha n}$$

for some $\alpha > \gamma$. Our next two results investigate, for a more general BRW, the subcases of $\gamma = 0$ in which

$$p_n(x,x) \sim c n^{-\delta} L(n),$$

where here and in the rest of the paper $L(n)$ denotes a slowly varying function. If this asymptotic form holds with arbitrary $\delta > 0$ on polynomially growing graphs, it is clear that all BRWs with almost surely exponentially growing families will be recurrent (regardless of the exponential parameter). In fact, we show the following.

Theorem 2 *If a BRW is almost surely exponentially growing on $G \in \mathcal{P}$ then it is recurrent.*

But this is far from the best possible. We obtain from the paper of Benjamini and Peres [BP1] on TIRWs that a polynomially growing family tree can be sufficient for recurrence if it grows quickly enough. Theorem 3 will demonstrate this. We need some definitions and remarks first.

Let A_n denote the size of Γ_n, the n-th level of the index tree Γ. In other words, this is the population size at time n (not the size of the n-th generation!) (c.f. [BP1]).

Suppose that $G \in \mathcal{P}_d$. Furthermore let us assume that the resistance growth has exponent d_Ω, the random walk has exponent d_R (sometimes denoted by d_w as well). These are some of the fundamental exponents describing RWs on fractals; see [RT], [T1] or [T2] for more details. We assume the notation of [T1] and [T2], and also include here the following notation on resistance of *finite* graphs. For $A, B \subset V$, $R(A, B)$ stands for the resistance between the two sets A and B if the edges of the graph are assumed to be unit resistors. We shall only study graphs with transient simple RWs on them, which means (c.f. [DS])

$$\lim_{N\to\infty} R_N = R_G < \infty.$$

In this case it is also useful to introduce the tail resistance

$$\rho_N = R_G - R_N$$

and remark that

$$\rho_N \sim N^{2-d_\Omega} L(N).$$

In [T1, Proposition 1] it is proved that

$$d \geq d_\Omega$$

if the limits in the definitions of the exponents can be achieved by sequences which do not grow very rapidly.

We define $T_{x,N}$ to be the hitting time of the sphere centred at x and of radius N by a particle starting at x.

Theorem 3 *Assume that $G \in \mathcal{P}$ and the following are true.*

(i) G is spherically symmetric about 0,

(ii) the mean escape time $E(T_{x,N})$ has uniform regular growth, i.e. there is are constants $c > 1$ and $d_R \geq 2$ such that

$$\frac{1}{c} N^{d_R} \leq E_x(T_{x,N}) \leq c N^{d_R}. \tag{1}$$

Then the TIRW is strongly recurrent if

$$\sum_n A_n n^{-1} \rho_N = \infty \tag{2}$$

where $N = cn^{\frac{1}{d_R}}$.

Remark 1

(a) [BP2, Proposition 6.1] states that a TIRW on a spherically symmetric graph is strongly recurrent if

$$\limsup_{n\to\infty} \frac{A_n}{|S_n|} > 0,$$

from which it follows that

$$\sum_{n=1}^{\infty} A_n n^{1-d} = \infty. \tag{3}$$

(b) [BP1, Theorem 1.3] basically states that for polynomially growing groups the corresponding TIRW is a.s. recurrent iff

$$\sum_n A_n n^{-\frac{d}{2}} = \infty. \tag{4}$$

This is equivalent to (2) for $d = d_\Omega$ and $d_R = 2$ (if all the exponents which appear exist) since (2) can be rewritten as

$$\sum_n A_n n^{-1} \rho_N = \sum_n A_n n^{\frac{2-d_\Omega}{d_R}-1} L(n^{\frac{1}{d_R}}) = \sum_n A_n n^{-\frac{d}{d_R}} L(n^{\frac{1}{d_R}})$$

if $d_R = d + 2 - d_\Omega$. There are several examples for graphs (fractals) of subdiffusive behaviour, that is, $d_R > 2$. A simple example is given in the next section.

Remark 2

Theorem 3 gives a better bound than (3) [BP2, Proposition 6.1] if $d > \frac{d_R}{d_R-1}$, and furthermore, provides the converse of the statement as well. Also, it gives a better result than (4) [BP1, Theorem 1.3] if $d_R > 2$, that is, in the subdiffusive cases.

Remark 3

The simple and nice character of TIRWs has a price. The tree (imitating a branching tree) does not depend on the RW. It would be very interesting to find a BRW with a "natural" branching mechanism which

- has a polynomially growing family of particles
- is recurrent iff (2) holds.

2 Examples

2.1 Population size dependent BRWs

From Klebaner [K], it is easy to construct a population size dependent BP of growth $n^{\frac{1}{1-\alpha}}$ for $\alpha \in (0,1)$. In this way one can obtain a BRW in which the BP is independent of the position of the particles, which means that the random family tree of the BP is an independent index tree for the RW, and Theorem 3 is applicable. With varying α the resulting BRW may be recurrent or transient. More detailed results can be obtained on particular fractals; a family of them is provided by the next example.

2.2 Sierpinski trees

The Sierpinski gasket provides the standard testing ground for new results on fractals. The classical SG(d') is recurrent. However, it is embedded in d' dimensions, and

$$d_\Omega = 2 - \frac{\log(d'+3) - \log d' + 1}{\log(2)} < 2.$$

In our investigation we need a fractal having a transient simple RW. We can define a simple one which exhibits the nice properties that SG(d') has, except recurrence.

Define a slowly growing symmetrical tree in the following way. (See also Doyle & Snell [DS].) From the root we have a branch of unit length which splits into q branches of length 2, and in general the k-th split produces q new branches of length 2^k at each vertex. In this way we have a tree with fractal (Hausdorff) dimension $\frac{\log q}{\log 2}$.

The next step is to replace a branch of length 2^k with a part of the Sierpinski gasket of the same length. The part we need can be constructed by m iterations of a procedure which begins with the d-simplex of unit side length, and at the ith iteration adds to the present configuration d copies shifted by 2^{i-1} (the shifts being in the directions parallel to the edges of the original simplex meeting at the "top" vertex). This part is joined in at two vertices of distance 2^k. Let us denote the resulting graph by $SG(d') \times T(q)$.

One can easily see that

$$d = \frac{\log((d'+1)q)}{\log 2}$$

and

$$d_\Omega = 2 - \frac{\log(d'+3) - \log((d'+1)q)}{\log 2},$$

which is greater than 2 if $q > \frac{d'+1}{d'+3}$, and so the RW is transient. Finally,

$$d_R = d + 2 - d_\Omega = \frac{\log(d'+3)}{\log 2}.$$

We note that this exponent does not change. Furthermore,

$$p_n(x,x) \sim n^{-\frac{d'}{d_R}} = n^{-\frac{\log((d'+1)q)}{\log(d'+3)}}.$$

Remark 4

One can construct $SG(d) \times T(q)$ for real $q > 1$ simply by alternating the k-th branching number $\lfloor q \rfloor$ and $\lceil q \rceil$ (lower or upper integer part) according as $|B_{2^k}| > q^k$ or $|B_{2^k}| < q^k$. The resulting dimensions will be as above.

2.3 Extreme-site birth rules

Consider the set of the particles $\mathcal{C}_n$ at a given time n and label the starting point 0. Let us define the "extreme" points g_n and f_n of $\mathcal{C}_n$ by choosing

$$g_n = x \quad \text{where} \quad d(0,x) = \min\{d(0,z) : z \in \mathcal{C}_n\}$$

and

$$f_n = y \quad \text{where} \quad d(0,y) = \max\{d(0,z) : z \in \mathcal{P}_n\}.$$

These extreme values can occur at more than one site but any of them is appropriate for our purposes (random uniform choice will do). New particles are produced only at the site g_n (then called the *closest birth* BRW) or f_n (then called *farthest birth* BRW) with family distribution P_k with mean m, for sake of simplicity not depending on other factors (e.g. the time n or the age of the particle at this site). Here we provide results only for Z^d, $d > 2$ and for P the point distribution at 1, but similar statements should be true on other graphs and also for more general birth distributions.

Proposition 1 *The closest birth BRW is strongly recurrent on Z^d for all d.*

Theorem 4 *The farthest birth BRW is recurrent on Z^d iff $d \leq 3$.*

2.4 The egg model

This model has several variations. We give here the simplest one which might be interesting. A single particle starts from 0, and initially there is one egg at each site. Any time a particle reaches a site with an egg on it, the egg turns into a new particle and starts its independent life.

Theorem 5 *The egg model is strongly recurrent on Z^d.*

3 Proofs

Proof of Theorem 1

Consider the pure site dependent BRW, with birth probabilities given by

$$f(k, x)$$

for all $x \in V$. By assumption there exists some point $x \in V$ with $f(0, x) \neq 1$. The recurrence of the BRW ensures the existence of an integer $n > 0$ such that

$$\sum_{i=1}^{i=n} \mathbf{P}(Z_i^x(x) > 0) > 0$$

where $Z_i^x(y)$ is the number of particles at $y \in V$ at time i started at $x \in V$. This means (recalling $f(0, x) < 1$) that the expected number of births at x up to time n is greater than 0. Hence the expected number of particles after n steps is strictly greater than 1, and the statement follows. ∎

We will need several times in the rest of the paper the following modification of one of the Borel-Cantelli lemmas. Its proof is virtually identical to the standard proof as in [Bi, Theorem 4.1], for example.

Lemma 1 *Let $A_1, A_2, \ldots$ be an infinite sequence of events from some probability space, and suppose that for all $n \geq 0$*

$$\sum_{j \geq n} \mathbf{P}(A_j \mid \bar{A}_n \cap \cdots \cap \bar{A}_{j-1}) = \infty.$$

Then with probability 1, infinitely many of the A_n occur.

Proof of Theorem 2

The resistance between S_n and infinity is at least $R(S_n, S_{n+1})$, which is at least the resistance obtained by shorting all the edges which do not join S_n to S_{n+1}. The polynomial growth of G ensures that there are $O(n^d)$ edges which do join S_n to S_{n+1} for some $d \geq 2$, and hence

$$R(S_n, \infty) > R(S_n, S_{n+1}) > cn^{-d}.$$

By assumption, the population Z_n of the BRW satisfies $|Z_n| > n^{d+1+\epsilon}$ for some $\epsilon > 0$ when n is sufficiently large, greater than some value n_1. Choose a set $F_1 \subseteq Z_{n_1}$ with $|F_1| = \lceil \frac{1}{2} n_1^{d+\epsilon} \rceil$. After n steps, all particles in the population are inside the ball B_n. The probability of return to the origin for such a particle is at least $R(S_n, \infty)$. So from the estimate above, the probability that at least one particle in F_1 eventually reaches the origin is at least $\frac{1}{2}$ if we initially choose n_1 sufficiently large. Similarly, with $n_i = in_1$ for $i > 1$, there exists an inductively defined sequence of sets $F_i \subseteq Z_{n_i}$ with $|F_i| = \lceil \frac{1}{2} n_i^{d+\epsilon} \rceil$, such that the probability that at least one particle in F_i eventually reaches the origin is at least $\frac{1}{2}$. Since $d \geq 2$, $\sum_{j=1}^{i-1} j^{d+\epsilon} < i^{d+1+\epsilon}$, and hence, in view of the exponential growth, the sets F_i can be chosen pairwise-disjoint. This immediately implies recurrence by Lemma 1. ∎

Proof of Theorem 3

We follow the idea of the proof of Proposition 6.1 in [BP2]. The proof uses the sequence $\tilde{S}_i = S_{2^i}$ of spheres for $i = 0, 1, \ldots$ and the abbreviations $\tilde{B}_i = B_{2^i}, n_i = c2^{d_R i}, \tilde{\Gamma}_i = \Gamma_{n_i}$. For all $\sigma \in \tilde{\Gamma}_i$ we pick a ray (an infinite path in Γ) containing σ, denoted by $\xi = \xi_\sigma \subset \bigcup_{k \geq n_i} \Gamma_k$.

The central object of the proof is a set of independent events defined for $\sigma \in \tilde{\Gamma}_i$ for $i \geq 0$, that is, the event Ω_σ that the particle represented by ξ_σ reaches $\tilde{S}_i$ and subsequently visits 0 before reaching $\tilde{S}_{i+1}$.

Since the rays are separate for $\sigma, \gamma \in \tilde{\Gamma}_i$, Ω_σ and Ω_γ are independent. In case of $\sigma \in \tilde{\Gamma}_i, \gamma \in \tilde{\Gamma}_j, i < j$ it is possible that $\gamma \in \xi_\sigma$, that is, $\xi_\gamma \subset \xi_\sigma$, but the event Ω_σ depends only upon the trajectory from T_i to T_{i+1} and Ω_γ from T_j to T_{j+1}, where T_k means the time when the particle along this ray reaches $T_{\tilde{S}_k}$. Since G is spherically symmetric, this means that all the Ω_σ-s are independent. Now

$$\sum_{i=0}^{\infty} \sum_{\sigma \in \tilde{\Gamma}_i} \mathbf{P}(\Omega_\sigma) \geq \sum_{i=0}^{\infty} \sum_{\sigma \in \tilde{\Gamma}_i} \mathbf{P}(\Omega_\sigma \mid T_i > n_i) \mathbf{P}(T_i > n_i).$$

The crucial difference from the proof given in [BP2, Proposition 6.1] is the estimate provided by [BP2, Lemma 6.2], is replaced by

$$\mathbf{P}(\Omega_\sigma \mid T_i > n_i) \geq c_3 \rho_{2^i}$$

which is a simple consequence of spherical symmetry and resistance calculation. Also, it is proved in [B] and [T] that given (1) there are $c, c^* > 0$ such that, for all $n > 0$

$$\mathbf{P}_x(T_{x,N} > n) > c^*$$

where $N = cn^{\frac{1}{d_R}}$ and $\mathbf{P}_x$ refers to a particle starting at x.

Using these inequalities and the monotonicity of A_n and ρ_N it follows that

$$\sum_{i=0}^{\infty} \sum_{\sigma \in \tilde{\Gamma}_i} \mathbf{P}(\Omega_\sigma) \geq \sum_{i=0}^{\infty} c_4 A_{n_i} \rho_{2^i} \geq \sum_{n=0}^{\infty} c_5 n^{-1} A_n \rho_{n^{1/d_R}}.$$

Strong recurrence now follows from the condition

$$\sum_{n=0}^{\infty} n^{-1} A_n \rho_{n^{1/d_R}} = \infty$$

and the Borel-Cantelli Lemma.

Proof of Proposition 1

The proof uses the observation that g_n has a positive drift toward the origin if it is not near the origin. Let us denote the individual born at time k by ξ_k and its position after it has taken s steps by $\xi_k(s)$. Consider a fixed even integer j and an arbitrary $x \in Z^d$, and assume that $g_n = x$ and that $d(0, x)$ is sufficiently large. Run the process for $j/2$ more steps. Consider the set U of $j/2$ newborn particles ξ_l for $l = n+1, \dots, n+j/2$. All of these are in $B_{x,j/2}$; furthermore, denoting $W_l = B_{d(0,x)-1} \cap B_{\xi_l(0),j}$ for such l, we have for $k = j^2 - l$, $y \in W_l$

$$\mathbf{P}(\xi_l(k) = y) \geq c_6 k^{-\frac{d}{2}} \geq c_7 j^{-d}.$$

Summing over W_l, we have

$$\mathbf{P}(\xi_l(k) \in B_{d(0,x)-1}) \geq |W_l| c_7 j^{-d} \geq c_8 =: 1 - q > 0.$$

In U there are $j/2$ independent particles, which ensure that

$$\mathbf{P}(g_{n+j^2} \in B_{d(0,x)-1}) \geq 1 - q^{\frac{j}{2}}$$

and thus the j^2 step drift towards the origin is at least $-j^2 q^{j/2} + 1(1 - q^{j/2})$.

This gives that if g_n starts from any given point other than 0, it can reach a finite neighbourhood of the origin with a number of steps linear in the distance, from where the origin can be reached with non-zero probability. The probability of this event can be bounded from below applying Markov's inequality to the result on the drift, and so the proposition follows from Lemma 1. ∎

Proof of Theorem 4

First some notation: we denote by $F(x, y)$ the probability that a particle starting from x ever reaches y.

Recurrence for $d \leq 2$ is immediate since the simple random walk is recurrent, and so, $F(x, 0) = 1$ for all x. For $d \geq 3$, we have

$$c_9 ||x||^{2-d} \geq F(x, 0) \geq c_{10} ||x||^{2-d} \tag{5}$$

for $x \in Z^d$ (c.f. (5.3) [BP1]). So for $d = 3$, given $f_0, f_1, \dots$ let us consider the events Φ_k, that ξ_k ever reaches 0. Then

$$\sum_{k \geq 0} \sum_k \mathbf{P}(\Phi_k) = \sum_{k \geq 0} F(f_k, 0) \mid X_0 = f_k) \geq \sum_k c_{10} k^{-1} = \infty.$$

which gives recurrence by Lemma 1. In fact, this shows strong recurrence.

We can now take $d \geq 4$. If the BRW is recurrent then infinitely many different particles have to enter the origin, since each of them is transient. We are interested in

$$\sum_{k=0}^{\infty} \mathbf{P}(\exists j > 0 : \xi_k(j) = 0) = \sum_{k=0}^{\infty} \mathbf{E}(F(\xi_k(0), 0))$$

where $F(x, y)$ denotes the probability of reaching y starting from x. Using the notation

$$A(k) = \{d(\xi_k(0), 0) > (c - \epsilon)k\}$$

for $0 < \epsilon < c$ (suitable c chosen by the constraints mentioned below), this summation can be written as

$$\sum_{k=0}^{\infty} \mathbf{E}(F(\xi_k(0), 0) \mid A(k))\mathbf{P}(A(k)) + \mathbf{E}(F(\xi_k(0), 0) \mid \overline{A(k)})\mathbf{P}(\overline{A(k)}). \qquad (6)$$

We need to establish the fact that f_n has a positive drift. This can be seen by considering the random $\{0.1. - 1\}$-variable $\delta = d(0, X_1) - d(0, X_0)$ of the simple RW $\langle X_i \rangle_{i \geq 0}$ for any $X_0 = x \in Z^d$. If δ_1 and δ_2 are two independent copies of δ then $\mathbf{E}(\max(\delta_1, \delta_2)) \geq c > 0$ for all $x \in Z^d$. The large deviation theorem for binomial events gives that

$$\mathbf{P}(\overline{A(k)}) < \exp(-\epsilon^2 k^2)$$

for a suitable value of c in the definition of $A(k)$. On the other hand, recall the upper bound on $F(x, 0)$ given in (5). Inserting these estimates into (6), we have

$$\sum_{k=0}^{\infty} \mathbf{P}(\exists j > 0 : \xi_k(j) = 0) \leq \sum_{k=0}^{\infty} \exp(-\epsilon^2 k^2) + c_{11} k^{2-d} < \infty$$

as $d > 3$ and hence the theorem, since the expected number of particles returning to the origin is finite. ∎

Proof of Theorem 5

Our proof is in the general setting of polynomially growing transient graphs whose exponents exist, except at two points, which we shall indicate, where we need the stronger condition in the theorem. We split our proof into three lemmas.

Lemma 2 *For a RW on Z^d starting at 0 the number r_N of sites visited in B_N is large in the sense that*

$$\mathbf{P}(r_N > N^{2-\epsilon}) > c_{12} > 0.$$

Lemma 3 *There exists an infinite sequence of non-intersecting nested annuli, and $\delta > 0$, such that for each annulus with outer radius r the volume is at least $c_{13}r^d$ and the proportion of eggs which are visited by at least one particle is at least $r^{-\delta}$.*

Let us call this the density property.

Lemma 4 *If there is an infinite sequence of annuli as in Lemma 3, then the BRW is strongly recurrent.*

Proof of Lemma 2

Note that the total trace of the RW in B_N is at least as large as the trace until the first exit time T_N:

$$r_N \geq \frac{1}{\max_{x\in B_N}\{l_x\}} \sum_{x\in B_N} l_x = \frac{1}{\max_{x\in B_N}\{l_x\}} T_N,$$

where l_x is the number of times the particle visits x until the first exit from B_N. At this point we need the integer lattice restriction for the first time to ensure that $F(x,x) < q < 1$. Thus

$$\mathbf{P}(l_x > N^\epsilon) < c_{14} q^{N^\epsilon}$$

and

$$\mathbf{P}\left(\max_{x\in B_N} l_x > N^\epsilon\right) < N^d c_{14} q^{N^\epsilon}$$

which is still small enough to give

$$\mathbf{P}(r_N > N^{d_R - \epsilon}) > c_{15} > 0.$$

The lemma follows since $d_R = 2$ for integer lattices. ∎

Proof of Lemma 3

This too follows from the monotonicity law on electric networks. For arbitrary N and for given $n = d(0,x)$

$$\begin{aligned}\mathbf{P}(T_0 < \infty \mid X_0 = x) &\geq \mathbf{P}(T_0 < T_N \mid X_0 = x) \\ &= \frac{R(\Pi_x, S_N)}{R_G - \rho_N} \\ &\geq c_G R(S_n, S_N) \\ &= c_G(\rho_n - \rho_N).\end{aligned}$$

Now let $N \to \infty$. ∎

Proof of Lemma 4

Let n be an arbitrary integer (later to be choosen suitably large). We prove the existence of the sequence of annuli in two steps. First we show by induction that at least one exists, of a bounded size, with fixed probability $c > 0$. Then it will be deduced that there is an infinite sequence of them with probability 1.

Step 1. Finding one annulus in which the walk is dense.

Let us denote the set of sites visited by a given set Ξ of particles by $\Re_\Xi$. Fix some $c > 2$ and for arbitrary n consider a sequence $n = r_1 < r_2 < \cdots < r_{d+3}$ where $4r_k > r_{k+1} > cr_k$, and consider the annulus defined by $\Lambda_k = B_{r_{2k+1}} - B_{r_{2k}}$. Note that $r_{d+3} < 4^{d+2}n$. Let ξ_j denote the j-th particle. From Lemma 2, with nonzero probability there are at least $n^{2-\epsilon}$ particles born in the ball B_n, as a result of the trajectory of ξ_1. This set of particles, which we call Ξ_1, is the initial set of particles for the induction.

We inductively define Ξ_{k+1} to be the set of particles born at the sites $\Re_{\Xi_k} \cap \Lambda_k$ and prove that

$$\mathbf{P}\left(|\Xi_{k+1}| > c_k n^{\min(k(2-\epsilon),d-\epsilon)}\right) > p_k$$

for appropriate constants p_k. The above statement verifies this for $k = 1$.

The inductive step works as follows. The starting set of particles is Ξ_k, which we assume without loss of generality are $\xi_1, \ldots, \xi_{|\Xi_k|}$. By induction, $|\Xi_k| > c_k n^{k(2-\epsilon)}$. We are interested in $|\Xi_{k+1}| = |\Re_{\Xi_k} \cap \Lambda_k|$.

Any given particle $\xi_j \in \Xi_k$ has a trajectory H_j in Λ_k after reaching the equator $s = (r_{2k} + r_{2k+1})/2$ in finite time at some site y. Using Lemma 2 for $B_{y,s-r_{2k}}$ we can deduce that

$$|H_j| \geq c_{16}(s - r_{2k})^{2-\epsilon} \geq c_{17}n^{2-\epsilon}$$

with positive probability. By using inclusion-exclusion (actually Bonferroni's inequality), we can write $|\Xi_{k+1}|$ as

$$|\Xi_{k+1}| \geq \sum_{j=1}^{|\Xi_k|} |H_j| - \sum_{x \in \Lambda_k} \sum_{j=1}^{|\Xi_k|} \sum_{l=j+1}^{|\Xi_k|} p_x(j) p_x(l) \tag{7}$$

where $p_x(j) = \mathbf{P}(x \text{ is visited by } \xi_j)$.

The first summand can be estimated in the following way. Conditional on the starting points of the particles in Ξ_k, the H_j are independent and thus from the comment above

$$\sum_{j=1}^{|\Xi_k|} |H_j| \geq |\Xi_k| c_{18} n^2 - \epsilon$$

with positive probability. We have to show that the second term in (7) is smaller than this. This is the second point where we need the integer lattice restriction. Using the estimate [BP1, (5.3)])

$$F(x,y) \leq c_{19} d(x,y)^{2-d},$$

and the fact that $d(x,y) > n$ for x the birth site of any $\xi_j \in \Xi_k$ and $y \in \Lambda_k$, we get that the expected value of the second term is smaller than

$$c_{20} |\Xi_k|^2 n^d n^{2(2-d)} = c_{20} |\Xi_k|^2 n^{4-d}.$$

Thus, by Markov's inequality, the second term is smaller than $c_{21}|\Xi_k|^2 n^{4-d+\epsilon}$ with probability $1 - c_{22} n^{-\epsilon}$. Thus, it is smaller than the first term with positive probability if

$$|\Xi_k| \leq c_{23} n^{d-2}.$$

Thus, at the last step, where $k = \lfloor d/2 \rfloor + 1$, we shall have

$$|\Xi_{k+1}| \geq c_{24} n^{d-\epsilon} \geq c_{25} r^{d-\epsilon}$$

with positive probability, where r is the outer radius of the last annulus used.

Step 2. The infinite sequence of annuli.

Let us consider an infinite sequence of nonintersecting annuli around 0, of the form defined in Step 1. Our method begins with n particles in the inner annulus and ensures the density property in one of the annuli, approximately the $d/2$-th annulus. Repeating Step 1 from the next annulus, we obtain a second annulus with the density property by considering annuli which do not intersect the annuli which the previous finite induction used. Repeating this gives that the density property holds for each of an infinite sequence of annuli independently with positive probability not depending on their size. Finally using Lemma 1 lemma we have with probability 1 an infinite series of annuli with the density property. ∎

Proof of Lemma 5

This is straightforward using Lemma 3. Given that cN^{d-2} new particles are born in a ball B_N, the probability that at least one visits the origin is greater than some constant if N is large enough by (5). Lemma 3 provides an infinite series of annuli with the required number of new born. Since the trajectories of the particles are independent given their birthsites, we can use Lemma 1 again and have that a.s. infinitely many particles will visit the origin. ∎

Remark 5

It follows similarly that with probability 1, all eggs are visited.

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