Growth Optimal Portfolio Selection Strategies with Transaction Cost

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Notation

investment in the stock market
d assets
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initial price $s_0^{(j)} = 1$, $j = 1, \ldots, d$

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$$x_n^{(j)} = \frac{s_n^{(j)}}{s_{n-1}^{(j)}}$$

$\mathbf{x}_n = (x_n^{(1)}, \ldots, x_n^{(d)})$ the return vector on trading period n

nth trading period a portfolio strategy

$$\mathbf{b}_n = (b_n^{(1)}, \ldots, b_n^{(d)}) = \mathbf{b}(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1})$$

$b_n^{(j)} \geq 0$ gives the proportion of the investor's capital invested in stock j for trading period n ($\sum_{j=1}^{d} b_n^{(j)} = 1$)

Portfolio selection without transaction cost

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for the nth trading period, S_{n-1} is the initial capital (it is invested).

$$S_n = S_{n-1} \sum_{j=1}^{d} b_n^{(j)} x_1^{(j)}$$

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with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \log \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle \, .$$

$$\frac{1}{n}\log S_n \quad \approx \quad \frac{1}{n}\sum_{i=1}^{n} \mathbf{E}\{\log \left\langle \mathbf{b}(\mathbf{X}_1^{i-1})\,,\,\mathbf{X}_i \right\rangle \mid \mathbf{X}_1^{i-1}\}$$

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and

$$\frac{1}{n}\log S_n^* \approx \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{\log\left\langle \mathbf{b}^*(\mathbf{X}_1^{i-1}),\mathbf{X}_i\right\rangle \mid \mathbf{X}_1^{i-1}\}$$

Portfolio selection with transaction cost

$S_0 = 1$, gross wealth S_n, net wealth N_n
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If $b_n^{(j)} x_n^{(j)} N_{n-1} \geq b_{n+1}^{(j)} N_n$ then we have to sell and the transaction cost at the j-th asset is

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otherwise we have to buy and the transaction cost at the j-th asset is

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$$\begin{aligned} N_n = S_n &- \sum_{j=1}^{d} c\left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n\right)^+ \\ &- \sum_{j=1}^{d} c\left(b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1}\right)^+, \end{aligned}$$

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$$\quad - \quad \sum_{j=1}^{d} c\left(b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1}\right)^+ ,$$

or equivalently

$$S_n = N_n \quad + \quad c\sum_{j=1}^{d}\left|b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n\right| .$$

Dividing both sides by S_n and introducing ratio

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we get

$$1 = w_n + c \sum_{j=1}^{d} \left| \frac{b_n^{(j)} x_n^{(j)}}{\langle \mathbf{b}_n \, , \, \mathbf{x}_n \rangle} - b_{n+1}^{(j)} w_n \right| .$$

$$S_n = N_{n-1}\langle \mathbf{b}_n \, , \, \mathbf{x}_n \rangle = S_{n-1} w_{n-1} \langle \mathbf{b}_n \, , \, \mathbf{x}_n \rangle = \prod_{i=1}^{n} [w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle]$$

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Introduce the notation

$$g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i) = \log(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle),$$

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then the average growth rate becomes

$$\begin{aligned} \frac{1}{n} \log S_n &= \frac{1}{n} \sum_{i=1}^{n} \log(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle) \\ &= \frac{1}{n} \sum_{i=1}^{n} g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i). \end{aligned}$$

In the sequel $\mathbf{x}_i$ will be random variable and is denoted by $\mathbf{X}_i$.
Let's use the decomposition

$$\begin{aligned} & \frac{1}{n}\log S_n \\ = \quad & \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}\{g(\mathbf{b}_{i-1},\mathbf{b}_i,\mathbf{X}_{i-1},\mathbf{X}_i)|\mathbf{X}_1^{i-1}\} \\ + \quad & \frac{1}{n}\sum_{i=1}^{n}(g(\mathbf{b}_{i-1},\mathbf{b}_i,\mathbf{X}_{i-1},\mathbf{X}_i)-\mathbf{E}\{g(\mathbf{b}_{i-1},\mathbf{b}_i,\mathbf{X}_{i-1},\mathbf{X}_i)|\mathbf{X}_1^{i-1}\}), \end{aligned}$$

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therefore

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If the market process $\{\mathbf{X}_i\}$ is a *homogeneous and first order Markov process* then

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therefore the maximization of the average growth rate

$$\frac{1}{n}\log S_n$$

is asymptotically equivalent to the maximization of

$$\frac{1}{n}\sum_{i=1}^{n} v(\mathbf{b}_{i-1},\mathbf{b}_i,\mathbf{X}_{i-1}).$$

dynamic programming problem

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empirical portfolio selection

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Define an infinite array of experts $\mathbf{B}^{(\ell)} = \{\mathbf{b}^{(\ell)}(\cdot)\}$, where ℓ is a positive integer.

For fixed positive integer ℓ, choose the radius $r_\ell > 0$ such that

$$\lim_{\ell \to \infty} r_\ell = 0.$$

put

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for $n > 1$, define the expert $\mathbf{b}^{(\ell)}$ by

$$\mathbf{b}_n^{(\ell)} = \arg\max_{\mathbf{b}\in\Delta_d} \sum_{\{i<n: \|\mathbf{x}_{i-1}-\mathbf{x}_{n-1}\|\leq r_\ell\}} \ln \langle \mathbf{b} \, , \, \mathbf{x}_i \rangle \quad ,$$

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and $\mathbf{b}_1 = (1/d, \ldots, 1/d)$ otherwise, where $\| \cdot \|$ denotes the Euclidean norm.

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the investor's capital is

$$S_n = S_{n-1}\langle \mathbf{b}_n \, , \, \mathbf{x}_n\rangle w(\mathbf{b}_{n-1}, \mathbf{b}_n, \mathbf{x}_{n-1}).$$

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These elementary portfolios are mixed as before (1) or (2).

NYSE data sets

At www.szit.bme.hu/~oti/portfolio there are two benchmark data set from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years (5651 trading days ending in 1985).
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Our experiment is on the second data set.

Kernel based log-optimal portfolio selection with $\ell = 1, \ldots, 10$

$$r_\ell^2 = 0.0001 \cdot d \cdot \ell,$$

Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with $\ell = 1, \dots, 10$

$$r_\ell^2 = 0.0001 \cdot d \cdot \ell,$$

MORRIS had the best AAY, 20%

The average annual yields of the individual experts and of the aggregations with $c = 0.0015$.

ℓ	$c = 0$	Algorithm 1	Algorithm 2
1	20%	-18%	-14%
2	118%	-2%	25%
3	71%	14%	55%
4	103%	28%	73%
5	134%	33%	77%
6	140%	43%	92%
7	148%	37%	83%
8	132%	38%	74%
9	127%	42%	66%
10	123%	44%	62%
Aggregation with wealth (1)	137%	**40%**	**83%**
Aggregation with portfolio (2)	137%	**49%**	**89%**

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discounted Bellman equation:

$$F_\delta(\mathbf{b}, \mathbf{x}) = \max_{\mathbf{b}'} \left\{ v(\mathbf{b}, \mathbf{b}', \mathbf{x}) + (1-\delta)\mathbf{E}\{F_\delta(\mathbf{b}', \mathbf{X}_2) \mid \mathbf{X}_1 = \mathbf{x}\} \right\}.$$

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Then, for Strategy 1, the portfolio $\{\mathbf{b}_i^*\}$ with capital S_n^* is optimal in the sense that for any portfolio strategy $\{\mathbf{b}_i\}$ with capital S_n,

$$\liminf_{n\to\infty} \left(\frac{1}{n} \log S_n^* - \frac{1}{n} \log S_n \right) \geq 0$$

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Choose an arbitrary probability distribution $q_k > 0$, and introduce the combined portfolio with its capital

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Then, for Strategy 2,

$$\lim_{n\to\infty} \left(\frac{1}{n} \log S_n^* - \frac{1}{n} \log \tilde{S}_n \right) = 0$$

a.s.

Open problem

How to construct empirical (data-driven) optimal portfolio selection strategy?