Principal component and constantly re-balanced portfolio

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September 20, 2007

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The model:

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Investment in the stock market: Growth rate

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Use the following simple bound

$$S_0 \max_j b^{(j)} S_n^{(j)} \leq S_n \leq d S_0 \max_j b^{(j)} S_n^{(j)}$$

assume that $b^{(j)} > 0$

$$\frac{1}{n}\ln \max_j \left(S_0 b^{(j)} S_n^{(j)}\right) \le \frac{1}{n}\ln S_n \le \frac{1}{n}\ln\left(d S_0 \max_j b^{(j)} S_n^{(j)}\right)$$

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$$\lim_{n\to\infty} \frac{1}{n} \ln S_n = \lim_{n\to\infty} \max_j \frac{1}{n} \ln S_n^{(j)} = \max_j W^{(j)}$$

Conclusion: any static portfolio achieves the maximal growth rate $\max_j W^{(j)}$.

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Conclusion: any static portfolio achieves the maximal growth rate $\max_j W^{(j)}$. We can do much better!

Dynamic portfolio selection: multi-period investment

The model:

- Let $\mathbf{x}_i = (x_i^{(1)}, \dots x_i^{(d)})$ the return vector on trading period i, where

$$x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}.$$

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This $\mathbf{b}$ is the portfolio vector for each trading day.

Dynamic portfolio selection: multi-period investment:2

Repeatedly investment:

- for the first trading period S_0 denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b} \, , \, \mathbf{x}_1 \rangle$$

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$$S_2 = S_1 \cdot \langle \mathbf{b}, \mathbf{x}_2 \rangle = S_0 \cdot \langle \mathbf{b}, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}, \mathbf{x}_2 \rangle .$$

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with the average growth rate

$$W_n(\mathbf{b}) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle \mathbf{b}, \mathbf{x}_i \rangle .$$

The CRP is the optimal portfolio for special market process, where $\mathbf{X}_1, \mathbf{X}_2, \ldots$ is independent and identically distributed (i.i.d.)

Log-optimum portfolio $\mathbf{b}^*$

$$\mathbf{E}\{\ln \langle \mathbf{b}^* , \mathbf{X}_1 \rangle\} = \max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b} , \mathbf{X}_1 \rangle\}$$

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Best Constantly Re-balanced Portfolio (BCRP)

Properties:

- needed full-knowledge on the distribution
- in experiments: not a causal strategy. We can calculate it only in hindsight.

Optimality

If $S_n^* = S_n(\mathbf{b}^*)$ denotes the capital after trading period n achieved by a log-optimum portfolio strategy $\mathbf{b}^*$, then for any portfolio strategy $\mathbf{b}$ with capital $S_n = S_n(\mathbf{b})$ and for any i.i.d. process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n \leq \lim_{n\to\infty} \frac{1}{n} \ln S_n^* \quad \text{almost surely}$$

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and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbf{E}\{\ln \langle \mathbf{b}^* , \mathbf{X}_1 \rangle\}$$

is the maximal growth rate of any portfolio.

Semi-log-optimal portfolio

log-optimal:

$$\arg\max_{\mathbf{b}} \mathbf{E}\{\ln \langle \mathbf{b} \,,\, \mathbf{X}_1 \rangle\}$$

It is a non-linear (convex) optimization problem with linear constraints. Calculation: not cheap.

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$$\ln z \approx h(z) = z - 1 - \frac{1}{2}(z-1)^2$$

Only the two biggest principal components, others are drop.

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Connection to the Markowitz theory.

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Gy. Ottucsák and I. Vajda, "An Asymptotic Analysis of the Mean-Variance portfolio selection", *Statistics&Decisions*, 25, pp. 63-88, 2007. http://www.szit.bme.hu/˜oti/portfolio/articles/marko.pdf

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then

$$\arg\max_{\mathbf{b}} -\frac{1}{2}\mathbf{E}\{(\langle \mathbf{b}\,,\,X_1\rangle - 2)^2\} + \frac{1}{2} = \arg\min_{\mathbf{b}} \mathbf{E}\{(\langle \mathbf{b}\,,\,X_1\rangle - 2)^2\},$$

that is, we are looking for the portfolio which minimize the expected squared error.

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  (go to Session 1 today at 17.30)
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

NYSE data sets

At `www.szit.bme.hu/~oti/portfolio` there are two benchmark data sets from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
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Our experiment is on the second data set.

Experimental results on CRP

Stock's name	AAY	BCRP	
		log-NLP weights	semi-log-QP weights
COMME	18%	0.3028	0.2962
HP	15%	0.0100	0.0317
KINAR	4%	0.2175	0.2130
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AAY		24%	24%
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KINAR had the smallest AAY

Dynamic portfolio selection: Causal-CRP

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we can even do much better!!

Dynamic portfolio selection: general case

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initial capital S_0

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for the second day, S_1 new initial capital, the portfolio vector $\mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1)$

$$S_2 = S_0 \cdot \langle \mathbf{b}_1 \, , \, \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1) \, , \, \mathbf{x}_2 \rangle \, .$$

nth day a portfolio strategy $\mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \dots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1})$

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle = S_0 e^{nW_n(\mathbf{B})}$$

with the average growth rate

$$W_n(\mathbf{B}) = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) \, , \, \mathbf{x}_i \right\rangle \, .$$

log-optimum portfolio

$\mathbf{X}_1, \mathbf{X}_2, \ldots$ drawn from the vector valued stationary and ergodic process
log-optimum portfolio $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$

$$\mathbf{E}\{\ln \left\langle \mathbf{b}^*(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) \, , \, \mathbf{X}_n \right\rangle \mid \mathbf{X}_1^{n-1}\}$$

$$\mathbf{X}_1^{n-1} = \mathbf{X}_1, \ldots, \mathbf{X}_{n-1}$$

Optimality

Algoet and Cover (1988): If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital after day n achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, then for any portfolio strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\limsup_{n\to\infty} \left(\frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}$$

for stationary ergodic process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$.

Kernel-based portfolio selection

fix integers $k, \ell = 1, 2, \dots$
elementary portfolios
choose the radius $r_{k,\ell} > 0$ such that for any fixed k,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$

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for $n > k+1$, define the expert $\mathbf{b}^{(k,\ell)}$ by

$$\mathbf{b}^{(k,\ell)}(\mathbf{x}_1^{n-1}) = \arg\max_{\mathbf{b}} \sum_{\left\{k<i<n: \|\mathbf{x}_{i-k}^{i-1} - \mathbf{x}_{n-k}^{n-1}\| \le r_{k,\ell}\right\}} \ln \langle \mathbf{b}\, , \, \mathbf{x}_i \rangle ,$$

if the sum is non-void, and $\mathbf{b}_0 = (1/d, \dots, 1/d)$ otherwise.

Combining elementary portfolios

let $\{q_{k,\ell}\}$ be a probability distribution on the set of all pairs (k, ℓ) such that for all k, ℓ, $q_{k,\ell} > 0$.

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The strategy $\mathbf{B}$ is the combination of the elementary portfolio strategies $\mathbf{B}^{(k,\ell)} = \{\mathbf{b}_n^{(k,\ell)}\}$ such that the investor's capital becomes

$$S_n(\mathbf{B}) = \sum_{k,\ell} q_{k,\ell} S_n(\mathbf{B}^{(k,\ell)}).$$

Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with
$k = 1, \ldots, 5$ and $\ell = 1, \ldots, 10$

$$r_{k,\ell}^2 = 0.0001 \cdot d \cdot k \cdot \ell,$$

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the BCRP had average AAY 24%

The average annual yields of the individual experts.

k / ℓ	1	2	3	4	5
1	20%	19%	16%	16%	16%
2	118%	77%	62%	24%	58%
3	71%	41%	26%	58%	21%
4	103%	94%	63%	97%	34%
5	134%	102%	100%	102%	67%
6	140%	125%	105%	108%	87%
7	148%	123%	107%	99%	96%
8	132%	112%	102%	85%	81%
9	127%	103%	98%	74%	72%
10	123%	92%	81%	65%	69%