Principal component and constantly re-balanced portfolio

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Investment in the stock market: Growth rate

The model:

- $d$ assets

\[
S_j(n) = S_j(0) e^{nW_j(n)}
\]

- $d$ average growth rate
- $W_j(n)$ asymptotic average growth rate

\[
W_j = \lim_{n \to \infty} \frac{1}{n} \ln S_j(n)
\]
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- $S_{n}^{(j)}$ price of asset $j$ at the end of trading period (day) $n$ initial price $S_{0}^{(j)} = 1$, 

\[ S_{n}^{(j)} = e^{n W(j)} \approx e^{n W(j)} \]

Asymptotic average growth rate

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Average growth rate

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asymptotic average growth rate

$$W^{(j)} = \lim_{n \to \infty} \frac{1}{n} \ln S_n^{(j)}$$
The aim is to achieve $\max_j W^{(j)}$. 
Static portfolio selection: single period investment

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Static portfolio selection:

- Fix a portfolio vector $\mathbf{b} = (b^{(1)}, \ldots, b^{(d)})$.
- $S_0 b^{(j)}$ denotes the proportion of the investor’s capital invested in asset $j$. Assumptions:
  - no short-sales $b^{(j)} \geq 0$
  - self-financing $\sum_j b^{(j)} = 1$
The aim is to achieve \( \max_j W^{(j)} \).

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After $n$ day

$$S_n = S_0 \sum_j b^{(j)} S_n^{(j)}$$

Use the following simple bound

$$S_0 \max_j b^{(j)} S_n^{(j)} \leq S_n \leq d S_0 \max_j b^{(j)} S_n^{(j)}$$
assume that $b^{(j)} > 0$

$$\frac{1}{n} \ln \max_j \left( S_0 b^{(j)} S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \frac{1}{n} \ln \left( dS_0 \max_j b^{(j)} S_n^{(j)} \right)$$

Conclusion: any static portfolio achieves the maximal growth rate $\max_j W^{(j)}$.

We can do much better!
assume that $b^{(j)} > 0$

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$$\max_{j} \left( \frac{1}{n} \ln S_0 b^{(j)} + \frac{1}{n} \ln S_n^{(j)} \right) \leq \frac{1}{n} \ln S_n \leq \max_{j} \left( \frac{1}{n} \ln (dS_0 b^{(j)}) + \frac{1}{n} \ln S_n^{(j)} \right)$$

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Dynamic portfolio selection: multi-period investment

The model:

- Let \( \mathbf{x}_i = (x_i^{(1)}, \ldots x_i^{(d)}) \) the return vector on trading period \( i \), where

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x_i^{(j)} = \frac{S_i^{(j)}}{S_{i-1}^{(j)}}.
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is the price relatives of two consecutive days.
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One of the simplest dynamic portfolio strategy is the Constantly Re-balanced Portfolio (CRP):

Fix a portfolio vector \( b = (b^{(1)}, \ldots, b^{(d)}) \), where \( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \).
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Fix a portfolio vector \( \mathbf{b} = (b^{(1)}, \ldots b^{(d)}) \), where \( b^{(j)} \) gives the proportion of the investor’s capital invested in stock \( j \).

This \( \mathbf{b} \) is the portfolio vector for each trading day.
Repeatedly investment:
- for the first trading period $S_0$ denotes the initial capital

$$S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle b, x_1 \rangle$$
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\[ S_1 = S_0 \sum_{j=1}^{d} b^{(j)} x_1^{(j)} = S_0 \langle b, x_1 \rangle \]

- for the second trading period, $S_1$ new initial capital

\[ S_2 = S_1 \cdot \langle b, x_2 \rangle = S_0 \cdot \langle b, x_1 \rangle \cdot \langle b, x_2 \rangle . \]
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- for the $n$th trading period:

$$S_n = S_{n-1} \langle \mathbf{b}, \mathbf{x}_n \rangle = S_0 \prod_{i=1}^{n} \langle \mathbf{b}, \mathbf{x}_i \rangle$$
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- for the $n$th trading period:
  \[ S_n = S_{n-1} \langle b, x_n \rangle = S_0 \prod_{i=1}^{n} \langle b, x_i \rangle = S_0 e^{nW_n(b)} \]
  with the average growth rate
  \[ W_n(b) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b, x_i \rangle. \]
The CRP is the optimal portfolio for special market process, where $X_1, X_2, \ldots$ is independent and identically distributed (i.i.d.)

Log-optimum portfolio $b^*$

$$E\{\ln \langle b^*, X_1 \rangle \} = \max_b E\{\ln \langle b, X_1 \rangle \}$$
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Best Constantly Re-balanced Portfolio (BCRP)
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Properties:
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Best Constantly Re-balanced Portfolio (BCRP)

Properties:

- needed full-knowledge on the distribution
- in experiments: not a causal strategy. We can calculate it only in hindsight.
If $S^*_n = S_n(b^*)$ denotes the capital after trading period $n$ achieved by a log-optimum portfolio strategy $b^*$, then for any portfolio strategy $b$ with capital $S_n = S_n(b)$ and for any i.i.d. process $\{X_n\}_{-\infty}^{\infty}$, 

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n \leq \lim_{n \to \infty} \frac{1}{n} \ln S^*_n \quad \text{almost surely}$$
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and

$$\lim_{n \to \infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely},$$

where

$$W^* = \mathbb{E}\{\ln \langle b^*, X_1 \rangle\}$$

is the maximal growth rate of any portfolio.
log-optimal:

$$\text{arg max}_b E\{\ln \langle b, X_1 \rangle\}$$

It is a non-linear (convex) optimization problem with linear constraints. Calculation: not cheap.
Semi-log-optimal portfolio

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Idea: use the Taylor expansion:

$$\ln z \approx h(z) = z - 1 - \frac{1}{2}(z - 1)^2$$

Only the two biggest principal components, others are drop.
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Connection to the Markowitz theory.
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$$E\{\langle b, X_1 \rangle - 1\} - \frac{1}{2} E\{(\langle b, X_1 \rangle - 1)^2\}$$
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\[
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= -\frac{1}{2} E\{(\langle b, X_1 \rangle - 2)^2\} + \frac{1}{2}
\]
We may write

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\[ = -\frac{1}{2} E\{(\langle b, X_1 \rangle - 2)^2\} + \frac{1}{2} \]

then

\[ \arg \max_b -\frac{1}{2} E\{(\langle b, X_1 \rangle - 2)^2\} + \frac{1}{2} = \]

\[ \arg \min_b E\{(\langle b, X_1 \rangle - 2)^2\}, \]

that is, we are looking for the portfolio which minimize the expected squared error.
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(go to Session 1 today at 17.30)
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- there are no transaction costs,
  (go to Session 1 today at 17.30)
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.
At www.szit.bme.hu/~oti/portfolio there are two benchmark data sets from NYSE:

- The first data set consists of daily data of 36 stocks with length 22 years.
- The second data set contains 23 stocks and has length 44 years.

Both sets are corrected with the dividends.
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Both sets are corrected with the dividends. Our experiment is on the second data set.
Experimental results on CRP

<table>
<thead>
<tr>
<th>Stock’s name</th>
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<th>BCRP</th>
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<tbody>
<tr>
<td></td>
<td></td>
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</tr>
<tr>
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<td>18%</td>
<td>0.3028</td>
</tr>
<tr>
<td>HP</td>
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<td>0.0100</td>
</tr>
<tr>
<td>KINAR</td>
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<td>0.2175</td>
</tr>
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Table: Comparison of the two algorithms for CRPs.
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KINAR had the smallest AAY
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we can even do much better!!
Dynamic portfolio selection: general case

\( \mathbf{x}_i = (x_i^{(1)}, \ldots x_i^{(d)}) \) the return vector on day \( i \)
\( \mathbf{b} = \mathbf{b}_1 \) is the portfolio vector for the first day
initial capital \( S_0 \)

\[
S_1 = S_0 \cdot \langle \mathbf{b}_1, \mathbf{x}_1 \rangle
\]
**Dynamic portfolio selection: general case**

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\[ S_1 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle \]

for the second day, \( S_1 \) new initial capital, the portfolio vector \( \mathbf{b}_2 = \mathbf{b}(\mathbf{x}_1) \)

\[ S_2 = S_0 \cdot \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle \cdot \langle \mathbf{b}(\mathbf{x}_1) , \mathbf{x}_2 \rangle . \]
Dynamic portfolio selection: general case

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\]

\( n \)th day a portfolio strategy \( \mathbf{b}_n = \mathbf{b}(\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}) = \mathbf{b}(\mathbf{x}_1^{n-1}) \)

\[
S_n = S_0 \prod_{i=1}^{n} \langle \mathbf{b}(\mathbf{x}_1^{i-1}), \mathbf{x}_i \rangle =
\]
Dynamic portfolio selection: general case

\[ x_i = (x_i^{(1)}, \ldots x_i^{(d)}) \]  the return vector on day \( i \)

\[ b = b_1 \]  is the portfolio vector for the first day

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\[ S_1 = S_0 \cdot \langle b_1, x_1 \rangle \]

for the second day, \( S_1 \) new initial capital, the portfolio vector

\[ b_2 = b(x_1) \]

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\( n \)th day a portfolio strategy \( b_n = b(x_1, \ldots, x_{n-1}) = b(x_1^{n-1}) \)

\[ S_n = S_0 \prod_{i=1}^{n} \langle b(x_1^{i-1}), x_i \rangle = S_0 e^{nW_n(B)} \]

with the average growth rate

\[ W_n(B) = \frac{1}{n} \sum_{i=1}^{n} \ln \langle b(x_1^{i-1}), x_i \rangle . \]
\( \mathbf{X}_1, \mathbf{X}_2, \ldots \) drawn from the vector valued stationary and ergodic process
log-optimum portfolio \( \mathbf{B}^* = \{\mathbf{b}^*(\cdot)\} \)

\[
\mathbf{E}\{\ln \langle \mathbf{b}^*(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \rangle | \mathbf{X}_1^{n-1} \} = \max_{\mathbf{b}(\cdot)} \mathbf{E}\{\ln \langle \mathbf{b}(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \rangle | \mathbf{X}_1^{n-1} \}
\]

\( \mathbf{X}_1^{n-1} = \mathbf{X}_1, \ldots, \mathbf{X}_{n-1} \)
Algoet and Cover (1988): If $S_n^* = S_n(B^*)$ denotes the capital after day $n$ achieved by a log-optimum portfolio strategy $B^*$, then for any portfolio strategy $B$ with capital $S_n = S_n(B)$ and for any process $\{X_n\}_{-\infty}^{\infty}$,

$$\limsup_{n \to \infty} \left( \frac{1}{n} \ln S_n - \frac{1}{n} \ln S_n^* \right) \leq 0 \quad \text{almost surely}$$

for stationary ergodic process $\{X_n\}_{-\infty}^{\infty}$. 
fix integers $k, \ell = 1, 2, \ldots$
elementary portfolios
choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$
fix integers $k, \ell = 1, 2, \ldots$

elementary portfolios

choose the radius $r_{k,\ell} > 0$ such that for any fixed $k$,

$$\lim_{\ell \to \infty} r_{k,\ell} = 0.$$ 

for $n > k + 1$, define the expert $b^{(k,\ell)}$ by

$$b^{(k,\ell)}(x_1^{n-1}) = \arg \max_{b} \sum_{\{k < i < n: \|x_{i-k}^{i-1} - x_{n-k}^{n-1}\| \leq r_{k,\ell}\}} \ln \langle b, x_i \rangle,$$

if the sum is non-void, and $b_0 = (1/d, \ldots, 1/d)$ otherwise.
Combining elementary portfolios

let \( \{q_{k,\ell}\} \) be a probability distribution on the set of all pairs \((k, \ell)\) such that for all \(k, \ell\), \(q_{k,\ell} > 0\).
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The strategy \( B \) is the combination of the elementary portfolio strategies \( B^{(k,\ell)} = \{b_{n}^{(k,\ell)}\} \) such that the investor’s capital becomes

\[
S_{n}(B) = \sum_{k,\ell} q_{k,\ell} S_{n}(B^{(k,\ell)}).
\]
Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with $k = 1, \ldots, 5$ and $\ell = 1, \ldots, 10$

$$r_{k,\ell}^2 = 0.0001 \cdot d \cdot k \cdot \ell,$$
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AAY of kernel based semi-log-optimal portfolio is 128%
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AAY of kernel based semi-log-optimal portfolio is 128% double the capital
MORRIS had the best AAY, 20%
Experiments on average annual yields (AAY)

Kernel based log-optimal portfolio selection with 
k = 1, \ldots, 5 \text{ and } \ell = 1, \ldots, 10

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AAY of kernel based semi-log-optimal portfolio is 128% double the capital
MORRIS had the best AAY, 20%
the BCRP had average AAY 24%
The average annual yields of the individual experts.

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