On Codecell Convexity of Optimal Multiresolution Scalar Quantizers for Continuous Sources

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***Abstract*—It has been shown by earlier results that for fixed rate multiresolution scalar quantizers and for mean squared error distortion measure, codecell convexity precludes optimality for certain discrete sources. However it was unknown whether the same phenomenon can occur for any continuous source. In this paper, examples of continuous sources (even with bounded continuous densities) are presented for which optimal fixed rate multiresolution scalar quantizers cannot have only convex codecells, proving that codecell convexity precludes optimality also for such regular sources.**

***Index Terms*—Clustering methods, codecell convexity, continuous density function, mean squared error methods, multiresolution, optimization methods, quantization, rate distortion theory, source coding**

I. Introduction and Definitions

Designing and studying quantizers with a given rate and minimum distortion is an important problem in data compression. In lossy IP and wireless network environments, the needs of robust communication implied increasing efforts on researching multiple description and multiresolution scalar quantization (MDSQ/MRSQ), a generalization of simple quantization. These concepts are defined formally below.

A. Scalar Quantizers

An *N-level scalar quantizer* (SQ) ($N \geq 1$ is an integer) is a measurable mapping $Q : \mathcal{R} \to \mathcal{C}$, where the *codebook* $\mathcal{C} = \{y_1, \dots, y_N\} \subset \mathcal{R}$ is a set of N (usually distinct) representation values, called the *code points*. The quantizer is completely characterized also by its codebook and the partition of the alphabet set $\mathcal{R}$ consisting of the N (usually nonempty) sets

$$C_i = \{x \in \mathcal{R} : Q(x) = y_i\}, \qquad i = 1, \dots, N$$

called the *partition cells* or *codecells* via the rule $Q(x) = y_i$, if $x \in C_i$. The set $\{C_1, \dots, C_N\}$ is called the *partition* of Q. A cell is said to be *convex* if it is a convex set, i.e., a contiguous interval of the real line. An SQ is called *convex* or *regular* if all of its cells are convex.

Let X be a random variable with distribution μ on the reals $\mathcal{R}$. The performance of the quantizer Q, when quantizing the *source* X, is measured by the *expected distortion* (or MSE)

$$D(\mu, Q) \stackrel{\text{def}}{=} \mathbb{E}\left\{(X - Q(X))^2\right\} = \sum_{i=1}^{N} \int_{C_i} (x - y_i)^2 \, \mu(dx). \tag{1}$$

We omit μ from the notation and write only $D(Q)$ where unambiguous. Throughout this paper, we assume $\mathbb{E}\left\{X^2\right\} < \infty$, thus, the distortion is finite.

A rate is associated to the quantizer, denoted by $R(Q)$. In the case of *fixed-rate* (FR) (or *resolution constrained*) quantizer, $R(Q) = \log_2 N$. The goal of optimal quantizer design is to minimize $D(Q)$ given a target quantizer rate $R(Q)$. We define the optimum distortion by

$$D^* \stackrel{\text{def}}{=} \inf_{Q \in \mathcal{Q}_N} D(Q)$$

where $\mathcal{Q}_N$ denotes the set of all N-level quantizers. If $D(Q^*) = D^*$ then $Q^* \in \mathcal{Q}_N$ is called *optimal*.

A quantizer Q satisfies the *nearest neighbor* or *optimal encoder condition* if

$$|x - Q(x)| = \min_{y_i \in \mathcal{C}} |x - y_i|, \qquad \text{for all } x \in \mathcal{R}.$$

Note that a nearest neighbor quantizer is determined by its codebook with ties arbitrarily broken and it is always convex. Note also that for any quantizer, a nearest neighbor quantizer with the same codebook has at most the same distortion regardless of μ. These imply (e.g., see [1]) that, for MSE fidelity criterion (and for single quantizers defined above), it suffices to consider nearest neighbor quantizers when searching for an optimal quantizer, and so finding an optimal quantizer is equivalent to finding its codebook. Using this, it is proved in [1] that there always exists an optimal quantizer, in particular, there is a convex optimal quantizer. Note that convexity has a central role in the arguments above.

B. Multiple Description and Multiresolution Scalar Quantizers

Multiple description and *multiresolution* (or *successively/progressively refinable*) *scalar quantizers* (MDSQ,MRSQ) are extensions of the quantizer model above. Following the line of [2], take M different SQs, Q_1, Q_2,

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$\ldots, Q_M$, called *side quantizers*, and define an M*-description scalar quantizer* (M-DSQ) $\mathbf{Q}$ to be a system of $2^M - 1$ SQs:

$$\{Q_{\mathcal{I}} : \emptyset \neq \mathcal{I} \subseteq \mathcal{M} = \{1, 2, \ldots, M\}\}$$

where for each $\mathcal{I} = \{i_1, \ldots, i_s\} \subseteq \mathcal{M}$, the partition of the *joint* or *component quantizer* $Q_{\mathcal{I}}$ is just the intersection of the partitions of $Q_{i_1}, \ldots, Q_{i_s}$, whereas its codebook is arbitrary. The expected distortion of $\mathbf{Q}$ is defined as

$$\bar{D}(\mu, \mathbf{Q}) \stackrel{\text{def}}{=} \sum_{\mathcal{I} \subseteq \mathcal{M}, \mathcal{I} \neq \emptyset} \omega_{\mathcal{I}} D(\mu, Q_{\mathcal{I}}) \tag{2}$$

where each $\mathcal{I} \subseteq \mathcal{M}$ is assigned a weight $\omega_{\mathcal{I}} \geq 0$, which practically represents the probability that only the side descriptions corresponding to $\mathcal{I}$ are available for source reconstruction in some networked source coding environment. (For more details on interpretations and motivations of this MDSQ model, see, e.g., [3], [2], [4], [5], [6]. The term for $\mathcal{I} = \emptyset$ is omitted since it does not affect the optimal design of MDSQ.) Again, we write only $\bar{D}(\mathbf{Q})$ where unambiguous.

When all M side quantizers are FR, the M-DSQ is said to be FR. We call quantizers $Q_{\mathcal{I}}$ with $\omega_{\mathcal{I}} \neq 0$, which contribute to the expected distortion, *active* components of the M-DSQ.

The above definition of MDSQ includes multiresolution scalar quantizers (MRSQ) as well, which have an additional prefix property. Precisely, an M*-resolution scalar quantizer* (M-RSQ) of M refinement stages is an M-DSQ whose active components are Q_1, $Q_{\{1,2\}}, \ldots, Q_{\{1,\ldots,i\}}, \ldots, Q_{\mathcal{M}}$.

The problem of optimal M-DSQ design is to minimize the expected distortion (2) over all possible M side quantizers, given the target rates $R_i \stackrel{\text{def}}{=} R(Q_i)$, $1 \leq i \leq M$ of the side quantizers and the weights $\omega_{\mathcal{I}}$. In particular, in optimal M-RSQ design, the weights $\omega_{\{1\}}, \omega_{\{1,2\}}, \ldots, \omega_{\{1,\ldots,i\}}, \ldots, \omega_{\mathcal{M}}$ are given. In the FR case, the constraint on the rate of Q_i is equivalent to have $N_i = 2^{R_i}$ cells.

Denote $\mathcal{Q}_{1,1}$ the set of all FR 2-DSQs (2-RSQs) with rates $R_1 = R_2 = 1$ (i.e., levels $N_1 = N_2 = 2$).

C. Convexity of MDSQs/MRSQs

An MDSQ/MRSQ is said to be *convex* if all its active quantizers are convex. (Thus, not all side quantizers are necessarily convex, but only if they are active. For example, in a convex MRSQ only the side quantizer Q_1 is active, therefore all the others may have nonconvex cells.)

Finding an optimal MDSQ/MRSQ is not easy in general. Although there are some MDSQ algorithms also for nonconvex cases (e.g., see [7], [8], [4]), several papers propose algorithms for convex MDSQs, and most MRSQ algorithms address the problem of finding the optimal one among convex MRSQs, that is, they hence lead to overall optimality if there is a convex optimal MDSQ/MRSQ (e.g., see [2], [9], [10], [11], [5], [12], [6], [13]). Under this constraint, optimal SQ design is equivalent to designing the optimal threshold sequence and corresponding codewords, so the search space is usually reduced significantly. Thus, it is a crucial question whether it suffices to consider convex MDSQs/MRSQs when searching for an optimal MDSQ/MRSQ.

A positive result is that for MSE distortion measure, cell convexity at the finest partition (i.e., that of $Q_{\mathcal{M}}$) can never preclude optimality, that is, the optimal distortion is always achievable using convex $Q_{\mathcal{M}}$. This is proven for discrete sources and MRSQs in [6, Theorem 6] and stated in general for MDSQs in [14], [6] (see also [7, Sec.III-A]).

However, unfortunately it turned out that this is not true for all other partitions, since convexity may preclude optimality: It is also shown in [14], [6] that there are discrete sources on finitely many atoms, target rates, and weights $\{\omega_{\mathcal{I}}\}$ such that the optimal MDSQ cannot be convex. Moreover, such counterexamples can be constructed even for some absolutely continuous sources. Namely, for the uniform distribution over $[0, 1]$ and for $\omega_{\{1\}} = \omega_{\{2\}}$, the optimal 2-DSQ in $\mathcal{Q}_{1,1}$ cannot be convex if $\omega_{\{1\}}/\omega_{\{1,2\}} < 7/81$, see also [2].

Convexity may preclude optimality for MRSQs, as well:

Proposition 1 ([14], [6]): There exists a (discrete) source on $\mathcal{R}$ and weights $\omega_{\{1\}}, \omega_{\{1,2\}} > 0$ such that the optimal 2-RSQ in $\mathcal{Q}_{1,1}$ cannot be convex.

However, as mentioned in [2], in case of (continuous) uniform source, the optimal MRSQ is always convex (since at each refinement stage the optimal quantizer must be the uniform quantizer with the corresponding rate $\sum_{j=1}^{i} R_j$). Thus the counterexample above for absolutely continuous sources does not work for MRSQs.

Also, related results show that for entropy constrained scalar quantizers (ECSQ, where $R(Q)$ is the entropy of the measure $\{\mu(C_i)\}_{i=1}^{N}$), there is also such (discrete) counterexample, however, as proven in [15], [16] in case of continuous sources and finite-level ECSQs, there is always a convex optimal ECSQ.

Considering these facts, one might think that perhaps also for MRSQs, the situation is different for discrete and continuous sources, that is, in case of, e.g., absolutely continuous sources over bounded regions with bounded probability density functions (pdf), there is always a convex optimal MRSQ. In this paper we want to clarify this question proving that this conjecture would fail as well. Our main results (Theorem 1, 2) show that for some weights and absolutely continuous sources with continuous pdfs, the optimal 2-RSQ in $\mathcal{Q}_{1,1}$ cannot be convex. Earlier results respecting to the class of continuous sources (e.g., [17] and [2]) also motivate the clarification of this case. One major technical difficulty of the proof for the continuous case is that the cell boundaries can split the mass of the source distribution anywhere. See Section II, for detailed motivation and difficulties.

Note that there are several other types of generalizations and extensions of the simple quantizer model and respecting results in Subsection I-A:

Distortion:	the fidelity criterion used in (1) can be different from MSE containing some other function of X and $Q(X)$.
VQ:	vector quantization, where the source alphabet is multidimensional.
VR, ECQ:	variable rate and (Rényi-)entropy constrained quantizers, where $R(Q) \neq \log_2 N$, but depends on the distribution $\{\mu(C_i)\}_{i=1}^{N}$.

The organization of the paper is as follows: In Section II, we give our main results along with their motivation in details. We present a lemma here that looks interesting on its own. Some technical lemmata and the proofs of our results are in Section III. In Section IV, we draw our conclusions and outline possible direction for future work. Finally, the proofs of the technical lemmata are given in the Appendix.

II. Main Result

Here we show that Proposition 1 remains true restricting ourselves to the case of absolutely continuous sources over bounded regions with bounded pdfs:

Theorem 1: There exists an absolutely continuous source μ with a bounded pdf over a bounded region of $\mathcal{R}$ and weights $\omega_{\{1\}}$,$\omega_{\{1,2\}} > 0$ such that the optimal 2-RSQ in $\mathcal{Q}_{1,1}$ for μ cannot be convex.

In Section III, we give a more quantitative version of this result (Lemma 5).

Note that clarifying the case of the class of (absolutely) continuous sources is required also because there are some results respecting to this important class. For example, it is proved in [17] and [2] that the local optimal convex MRSQ is unique (and thus search algorithms, as generalized Lloyd and gradient methods, always find the global optima correctly) if the source has a log-concave pdf (consequently it is absolutely continuous). In [2], also there is a supporting argument that high resolution MRSQs are convex. This argument is based on compander functions, and so applies, first of all, to continuous sources.

Using a quite elegant continuity argument we can develop a counterexample with continuous pdf based on one with piece-wise constant pdf as in the proof of Theorem 1. Theorem 2, a sharpened version of Theorem 1, and Lemmata 1 and 7 below show this:

Theorem 2: There exists an absolutely continuous source μ with a bounded continuous pdf over a bounded region of $\mathcal{R}$ and weights $\omega_{\{1\}}$, $\omega_{\{1,2\}} > 0$ such that the optimal 2-RSQ in $\mathcal{Q}_{1,1}$ for μ cannot be convex.

In Section III, we give a more quantitative version of this result, too (Lemma 7).

For the proof, we use the following noteworthy observation: If a piecewise constant pdf is approximated by a continuous pdf then the distortion of an SQ for the former is uniformly approximated by the distortion for the later. To make this precise, let f be a piecewise constant pdf supported on K disjoint closed intervals in $[0,1]$:

$$f = \sum_{i=1}^{K} a_i \mathbb{I}_{\{[b_i, b_i+w_i]\}} \tag{3}$$

where $b_i, w_i > 0$, $b_i + w_i < b_{i+1} < 1$, $b_K + w_K < 1$. For

$$0 < \epsilon < \min\left(b_1, 1 - b_K - w_K, \min_{1\le i\le K-1} \frac{b_{i+1} - b_i - w_i}{2}, \min_{1\le i\le K} \frac{w_i}{2}\right) \tag{4}$$

let f_ϵ be a continuous approximation of f replacing its jump discontinuities by linear pieces on 2ϵ wide intervals:

$$\begin{aligned} f_\epsilon(x) = \sum_{i=1}^{K} \Big(& \frac{a_i}{2\epsilon}(x - b_i + \epsilon)\mathbb{I}_{\{|x-b_i|\le\epsilon\}} \\ & + a_i \mathbb{I}_{\{x\in[b_i+\epsilon, b_i+w_i-\epsilon]\}} \\ & + \frac{a_i}{2\epsilon}(b_i + w_i + \epsilon - x)\mathbb{I}_{\{|x-b_i-w_i|\le\epsilon\}}\Big). \end{aligned} \tag{5}$$

Then the observation is formalized as shown.

Lemma 1: Let μ and μ_ϵ be the sources corresponding to f in (3) and f_ϵ in (5), respectively. Whenever (4) holds for ϵ, then for any SQ Q with all code points in $[0,1]$, we have

$$|D(\mu_\epsilon, Q) - D(\mu, Q)| \le \epsilon \sum_{i=1}^{K} a_i.$$

We need the following concept as well: A quantizer Q satisfies the *centroid* or *optimal decoder condition* for a given source X if each code point is chosen to minimize the distortion over its associated cell, that is, for the MSE fidelity criterion

$$\mathbb{E}\left\{(X - y_i)^2 | X \in C_i\right\} = \min_{y\in\mathcal{R}} \mathbb{E}\left\{(X - y)^2 | X \in C_i\right\} \tag{6}$$

and so $y_i = \mathbb{E}\{X | X \in C_i\}$ for all $i = 1, \ldots, N$. Note that such a quantizer is determined by its partition (given the source). It is well known (e.g., see [18], [19]) that any optimal quantizer satisfies the centroid condition.

Note that any active component of an optimal MDSQ/MRSQ satisfies the centroid condition, but not necessarily the nearest neighbor condition.

III. Proofs

The following notations for a 2-RSQ $\mathbf{Q}$ (with active components Q_1 and $Q_{\{1,2\}}$) will be useful: $D_1(\mathbf{Q}) \stackrel{\text{def}}{=} D(Q_1)$, $D_{\{1,2\}}(\mathbf{Q}) \stackrel{\text{def}}{=} D(Q_{\{1,2\}})$, $\omega \stackrel{\text{def}}{=} \omega_{\{1\}}/\omega_{\{1,2\}}$.

In the proof of Proposition 1, it is shown that for the discrete distribution with probability mass function $\{1/8, 1/8, 3/8, 3/8\}$ on alphabet $\{2, 4, 6, 14\}$, when ω is small enough, an optimal 2-RSQ $\mathbf{Q}$ cannot be convex. For absolutely continuous sources, we can use a continuous distribution replacing the atoms by intervals of a pdf with length h approximating the atomic distribution above as $h \ll 1$. Why is this case much more involved to handle then? The discrete source case is based on the observation that since the cell partitions of a quantizer are "forced to decide" which atom belongs to which cell, the amount of the suboptimality (distortion redundancy) of $Q_{\{1,2\}}$ is discrete, thus it could overwhelm any change in $D_1(\mathbf{Q})$ when ω is small, and so the optimality of the 2-RSQ forces the optimality of $Q_{\{1,2\}}$, that is, $D_{\{1,2\}}(\mathbf{Q}) = 0$. Then $D_1(\mathbf{Q})$ is optimized by a noncontiguous partitioning of $\mathcal{R}$. For absolutely continuous sources, the cell boundaries can split the intervals of the pdf anywhere. Thus for example, the amount of suboptimality of $Q_{\{1,2\}}$ can be arbitrarily tiny, so the discussion have to exclude such cases by other means. Another approach to see the difficulty is the following: Consider the rate-distortion (R-D) domain defined by

$$\{(R_1, R_2, D_1(\mathbf{Q}), D_{\{1,2\}}(\mathbf{Q})) \in \mathcal{R}^4 : \mathbf{Q} \in \{\text{2-RSQs}\}\}$$

where $\mathbf{Q}$ runs over the set of all 2-RSQs. For the discrete source that was used as counterexample in Proposition 1, we can see that the point $(1, 1, 10.875, 0) \in \mathcal{R}^4$ is on the boundary of the R-D domain. Moreover, because of the 0 distortion at the finer resolution, it is quite easy to see that this point is on the lower convex hull of the R-D domain implying that the corresponding (nonconvex) 2-RSQ also minimizes some weighted average distortion like (2) (see [14] and [6]). Furthermore, due to the discrete source, the R-D domain must have a jump (a corner) at this point, thus it is easy to show that there are weights for which this 2-RSQ is the only optimal one. These facts give the basis for the precise proof of Proposition 1. For the continuous counterpart source, as in the proof of Theorem 1, we have positive distortion at the finer resolution instead, and it is not obvious too see whether the corresponding point is on the lower convex hull of the R-D domain. Moreover, if it is there, there is no jump in the R-D domain due to continuity, and the uniqueness of the optimal 2-RSQ is still more involved to show.

One could think that some simple continuity argument (e.g., similar to Lemma 1) can be used saying that the distortion of any quantizer on the discrete counterexample and on the approximating absolutely continuous counterexample corresponding to h (the measure of approximation, see (7) below) cannot differ more than, let us say, $O(h)$ as $h \to 0$. For most of the quantizers, this holds (even with $O(h^2)$) quantizer-wise, but it does not hold uniformly, that is, the threshold for h depends on the quantizer and can be arbitrarily small. Thus this argument requires an involved discussion on deferent cases, which is basically done in our proof below.

We need the following lemmata proved in the Appendix.

Lemma 2: Let

$$f = \mathbb{I}_{\{[-1-2w_1,-1]\}} a_1 + \mathbb{I}_{\{[1,1+2w_2]\}} a_2.$$

If a quantizer cell contains the support of f, the pdf of the source is given by (or proportional to) f, and the corresponding code point is positioned optimally, then the optimal code point in the cell is

$$\frac{a_2 w_2(1 + w_2) - a_1 w_1(1 + w_1)}{a_1 w_1 + a_2 w_2}.$$

Lemma 3: Let

$$f = \mathbb{I}_{\{[-1-2w_1 h,-1]\}} a_1/h + \mathbb{I}_{\{[1,1+2w_2 h]\}} a_2/h.$$

If a quantizer cell contains the support of f and the pdf of the source is given by f, then the distortion contribution of the cell is

$$d_{h,w_1,a_1,w_2,a_2} = \frac{2}{3}\frac{a_1 a_2 w_1 w_2(12 + 6s + s^2) + h^2(a_2 w_2^2 - a_1 w_1^2)^2}{a_1 w_1 + a_2 w_2}$$

where $s = 2h(w_1 + w_2)$ the length of the support of f. In particular,

$$d_{h,w_1,a_1,0,a_2} = d_{h,w_1,a_1,w_2,0} = 2h^2 a_1 w_1^3/3 \qquad \text{and}$$
$$d_{h,w,a,w,a} = aw(4 + 2s + s^2/3).$$

Moreover, if the pdf in the cell is lower bounded by f, then the distortion contribution of the cell is at least d_{h,w_1,a_1,w_2,a_2}.

Lemma 4: Let Q be an SQ such that its cell C mapped to code point y contains the interval $[a, b]$ which has positive μ-measure. Let Q' be the SQ with the same codebook and partition as Q except that $[a, b]$ is transposed from C to $\tilde{C}$ (mapped to $\tilde{y}$), that is, $C' = C \setminus [a, b]$ and $\tilde{C}' = \tilde{C} \cup [a, b]$. If
i) $y < \tilde{y}$ and $y + \tilde{y} < 2a$ or
ii) $y > \tilde{y}$ and $y + \tilde{y} > 2b$,
then $D(Q) > D(Q')$. If $\mu([a, b]) = 0$ then $D(Q) = D(Q')$. Moreover, if Q'' is the SQ with the same partition as Q' but satisfying the centroid condition, then $D(Q') \geq D(Q'')$.

We prove Theorem 1 approximating the finite-source counterexample in [14] and [6] by an absolutely continuous one. Let $h > 0$ and consider the distribution μ having pdf

$$f = \frac{1}{16h}\big(\mathbb{I}_{\{x\in[2-4h,2-2h]\}} + \mathbb{I}_{\{x\in[4-2h,4]\}} + 3\mathbb{I}_{\{x\in[6,6+2h]\}} + 3\mathbb{I}_{\{x\in[14+2h,14+4h]\}}\big) \tag{7}$$

supported on four disjoint intervals with weighted uniform distributions on each of them. The following lemma gives a more technical version of Theorem 1:

Lemma 5: If $\omega \leq 1/300$ and $h \leq 3/16$ then for the source μ corresponding to f in (7) above and for any convex 2-RSQ $\mathbf{Q} \in \mathcal{Q}_{1,1}$

$$\bar{D}(\mathbf{Q}) - \inf_{\mathbf{Q}'\in\mathcal{Q}_{1,1}} \bar{D}(\mathbf{Q}') \geq \omega_{\{1\}}(0.697 - 4.472h - 2.07h^2).$$

Proof of Lemma 5: By (2), the distortion of a 2-RSQ is

$$\bar{D}(\mathbf{Q}) = \omega_{\{1\}} D_1(\mathbf{Q}) + \omega_{\{1,2\}} D_{\{1,2\}}(\mathbf{Q})$$
$$= \omega_{\{1,2\}}(\omega D_1(\mathbf{Q}) + D_{\{1,2\}}(\mathbf{Q})).$$

For concision, we assume without loss of generality (w.l.o.g.) that the scaling factor $\omega_{\{1,2\}} = 1$ and so $\omega = \omega_{\{1\}}$.

Note that, since the distortion of a convex MRSQ on μ is a continuous function of the cell boundaries and the code points, it is easily seen from Weierstrass' Extreme value theorem that it takes its minimum, that is, there is an optimal one among all convex MRSQs with given rates. So we can assume w.l.o.g. that $\mathbf{Q}$ is optimal for μ and $\omega_{\{1\}}$, $\omega_{\{1,2\}}$ among all convex 2-RSQs in $\mathcal{Q}_{1,1}$.

Convexity of $\mathbf{Q}$ and $N_1 = N_2 = 2$ imply that its first partition consists of two half-lines, and its second (refined) partition splits both into an interval and a half-line. Formally, for appropriate t_1, t_2, t_3,

$$P_1 = \{(-\infty, t_2), [t_2, \infty)\} \qquad \text{and}$$
$$P_{\{1,2\}} = \{(-\infty, t_1), [t_1, t_2), [t_2, t_3), [t_3, \infty)\}.$$

(For continuous distributions, it is insignificant to which cells the boundary points t_i belong.) The optimality of $\mathbf{Q}$ implies that Q_1 and $Q_{\{1,2\}}$, the active components of $\mathbf{Q}$, satisfy the centroid condition, which determines their code points $\{y_1, y_2\}$ and $\{c_1, c_2, c_3, c_4\}$, respectively based on $\{t_i\}_{i=1}^3$.

Define the "trivial" convex 2-RSQ $\mathbf{Q}^0$ as an instance of $\mathbf{Q}$ when $t_1 = 4-2h$, $t_2 = 6$, $t_3 = 14+2h$ (i.e., each cell of the refined partition contains exactly one interval of the support and the centroid condition is satisfied). Recalling the definition of d_{h,w_1,a_1,w_2,a_2}, by Lemma 3, its corresponding distortions are

$$\begin{aligned} D_1(\mathbf{Q}^0) &= d_{h,1,1/16,1,1/16} + 64 d_{h,1/4,3/16,1/4,3/16} \\ &= \frac{49}{4} + 6.5h + \frac{4h^2}{3} \qquad \text{and} \\ D_{\{1,2\}}(\mathbf{Q}^0) &= d_{h,1,1/16,0,0} + d_{h,1,1/16,0,0} \\ &\quad + d_{h,1,3/16,0,0} + d_{h,1,3/16,0,0} \\ &= \frac{h^2}{3} \end{aligned}$$

and thus,

$$\begin{aligned} \bar{D}(\mathbf{Q}^0) &= \omega D_1(\mathbf{Q}^0) + D_{\{1,2\}}(\mathbf{Q}^0) \\ &= \omega\left(\frac{49}{4} + 6.5h + \frac{4h^2}{3}\right) + \frac{h^2}{3}. \end{aligned}$$

Note that this and $\omega \le 1/300$ imply

$$\bar{D}(\mathbf{Q}^0) \le \frac{1}{300}\left(\frac{49}{4} + \frac{13h}{2} + \frac{304h^2}{3}\right). \tag{8}$$

Also we have the following lemma (that is obvious in the limit $h \to 0$) proven in the Appendix:

Lemma 6: If $h \le 3/16$ then $Q^0_{\{1,2\}}$ minimizes $D(\mu, Q)$ over all 4-level (2-rate) quantizers Q.

Case $t_2 < 6$: It is proven in the Appendix that now $D_1(\mathbf{Q}^0) \le D_1(\mathbf{Q})$. By Lemma 6 also $D_{\{1,2\}}(\mathbf{Q}^0) \le D_{\{1,2\}}(\mathbf{Q})$, thus $\bar{D}(\mathbf{Q}^0) \le \bar{D}(\mathbf{Q})$, hence $\mathbf{Q}^0$ must also be optimal convex 2-RSQ, which is covered by the third case below.

Case $t_2 > 6 + 0.1h$: Now, if $t_1 \le 4 - 1.8h$ then considering the interval $[4-1.8h, 6+0.1h]$, $D_{\{1,2\}}(\mathbf{Q}) > d_{h,0.9,1/16,0.05,3/16}$, whereas if $t_1 > 4-1.8h$ then considering the interval $[2-4h, 4-1.8h]$, $D_{\{1,2\}}(\mathbf{Q}) > d_{h,1,1/16,0.1,1/16}$, giving together anyway

$$D_{\{1,2\}}(\mathbf{Q}) > \min(d_{h,0.9,1/16,0.05,3/16}, d_{h,1,1/16,0.1,1/16}).$$

Substituting by Lemma 3, this is

$$\begin{aligned} \min&\left(\frac{18 + 17.1h + 12.570\,625h^2}{280}, \frac{30 + 33h + 36.6025h^2}{660}\right) \\ &> \frac{1}{300}\left(\frac{49}{4} + \frac{13h}{2} + \frac{304h^2}{3}\right) \end{aligned}$$

(from $h \le 3/16$), which in turn at least $\bar{D}(\mathbf{Q}^0)$ by (8). So in this case, $\bar{D}(\mathbf{Q}) > \bar{D}(\mathbf{Q}^0)$, that is, $\mathbf{Q}$ cannot be optimal convex 2-RSQ.

Case $t_2 \in [6, 6+0.1h]$: Let $\epsilon_2 \overset{\text{def}}{=} (t_2 - 6)/2 \in [0, h/20]$. Now, for a given t_2, determine the optimal values of t_1 and t_3. These do not influence $D_1(\mathbf{Q})$, thus their choices have to minimize $D_{\{1,2\}}(\mathbf{Q})$.

If $(t_2 \le) t_3 < 6+2h$ held, moving the boundary from t_3 into $6+2h$ and applying Lemma 4 case ii) with SQ $Q_{\{1,2\}}$, interval $[t_3, 6+2h]$, and code points (c_4, c_3), we would get a convex 2-RSQ with smaller distortion. Then if $6+2h \le t_3 < 14+2h$ held, increasing t_3 would not change $D_{\{1,2\}}(\mathbf{Q})$, whereas if $t_3 > 14+4h$ held, decreasing t_3 would not change $D_{\{1,2\}}(\mathbf{Q})$. Thus w.l.o.g. we can assume that $t_3 \in [14+2h, 14+4h]$. Let $\epsilon_3 \overset{\text{def}}{=} (t_3 - 14 - 2h)/2 \in [0, h]$. For a given t_3, c_3 and c_4 are determined by the centroid condition:

$$\begin{aligned} c_3 &= \frac{2(h-\epsilon_2)(6+h+\epsilon_2) + 2\epsilon_3(14+2h+\epsilon_3)}{2(h-\epsilon_2) + 2\epsilon_3} \\ &= 6 + h + \epsilon_2 + \epsilon_3 + \frac{8\epsilon_3}{h - \epsilon_2 + \epsilon_3} \\ c_4 &= 14 + 3h + \epsilon_3. \end{aligned}$$

These imply that

$$\begin{aligned} c_3 + c_4 &= 20 + 4h + \epsilon_2 + 2\epsilon_3 + \frac{8}{1 + (h-\epsilon_2)/\epsilon_3} \\ &\le 20 + 4h + \frac{h}{20} + 2h + \frac{8}{1 + 19/20} \\ &= 20 + 4h + \frac{41h}{20} + \frac{160}{39} \\ &< 2(14+2h) \qquad \text{(from } h \le 3/16\text{)}. \end{aligned}$$

So if we had $t_3 > 14+2h$, then moving the boundary from t_3 into $14+2h$ and applying Lemma 4 case i) with SQ $Q_{\{1,2\}}$, interval $[14+2h, t_3]$, and code points (c_3, c_4), we would get a convex 2-RSQ with smaller distortion. Thus $\mathbf{Q}$ cannot be optimal convex 2-RSQ unless $t_3 = 14+2h$.

If $t_1 \le 2-4h$ held, moving t_1 into $2 = 4h$ would not change $D_{\{1,2\}}(\mathbf{Q})$. Then if $2-4h < t_1 < 2-2h$ held, moving the boundary from t_1 into $2-2h$ and applying Lemma 4 case ii) with SQ $Q_{\{1,2\}}$, interval $[t_1, 2-2h]$, and code points (c_2, c_1), we would get a convex 2-RSQ with smaller distortion. Then if $2-2h \le t_1 < 4-2h$ held, increasing t_1 would not change $D_{\{1,2\}}(\mathbf{Q})$. If $6 < t_1 (\le t_2)$ held, moving the boundary from t_1 into 6 and applying Lemma 4 case i) with SQ $Q_{\{1,2\}}$, interval $[6, t_1]$, and code points (c_1, c_2), we would get a convex 2-RSQ with smaller distortion. Then if $4 < t_1 \le 6$ held, decreasing t_1 would not change $D_{\{1,2\}}(\mathbf{Q})$). Thus w.l.o.g. we can assume that $t_1 \in [4-2h, 4]$. Moreover, if $t_1 > 4-h$ held, then considering the interval $[2-4h, 4-h]$, we would have (using Lemma 3)

$$\begin{aligned} D_{\{1,2\}}(\mathbf{Q}) > d_{h,1,1/16,1/2,1/16} &= \frac{1}{6} + \frac{h}{4} + \frac{9h^2}{64} \\ &> \frac{1}{300}\left(\frac{49}{4} + \frac{13h}{2} + \frac{304h^2}{3}\right) \end{aligned}$$

(from $h \le 3/16$), which in turn at least $\bar{D}(\mathbf{Q}^0)$ by (8). So in this case, $\bar{D}(\mathbf{Q}) > \bar{D}(\mathbf{Q}^0)$, that is, $\mathbf{Q}$ could not be optimal convex 2-RSQ. Hence w.l.o.g. we can assume that $t_1 \in [4-2h, 4-h]$. Let $\epsilon_1 \overset{\text{def}}{=} (t_1 - 4 + 2h)/2 \in [0, h/2]$. For a given t_1, c_1 and c_2 are determined by the centroid condition:

$$\begin{aligned} c_1 &= \frac{2h(2-2h-h) + 2\epsilon_1(4-2h+\epsilon_1)}{2h + 2\epsilon_1} \\ &= 2 - 3h + \epsilon_1 + \frac{2\epsilon_1}{h + \epsilon_1} \end{aligned}$$

$$\begin{aligned} c_2 &= \frac{2(h-\epsilon_1)(4-2h+h+\epsilon_1) + 3\cdot 2\epsilon_2(6+\epsilon_2)}{2(h-\epsilon_1) + 3\cdot 2\epsilon_2} \\ &= 4 - h + \epsilon_1 + 3\epsilon_2 + \frac{6\epsilon_2(1-\epsilon_2)}{h - \epsilon_1 + 3\epsilon_2}. \end{aligned}$$

These imply that

$$\begin{aligned} &c_1 + c_2 \\ &= 6 - 4h + 2\epsilon_1 + 3\epsilon_2 + \frac{2}{1+h/\epsilon_1} + \frac{6(1-\epsilon_2)}{3+(h-\epsilon_1)/\epsilon_2} \\ &\le 6 - 4h + h + \frac{3h}{20} + \frac{2}{1+2} + \frac{6}{3+20/2} \\ &= 6 - 4h + \frac{23h}{20} + \frac{44}{39} \\ &< 2(4-2h) \qquad \text{(from } h \le 3/16\text{)}. \end{aligned}$$

So if we had $t_1 > 4-2h$, then moving the boundary from t_1 into $4-2h$ and applying Lemma 4 case i) with SQ $Q_{\{1,2\}}$, interval $[4-2h, t_1]$, and code points (c_1, c_2), we would get a convex 2-RSQ with smaller distortion. Thus $\mathbf{Q}$ cannot be optimal convex 2-RSQ unless $t_1 = 4-2h$.

Thus the optima can be only in $t_1 = 4-2h$ and $t_3 = 14+2h$ in this case. W.l.o.g. we assume these from now on.

Define $\mathbf{Q}' \in \mathcal{Q}_{1,1}$ as a 2-RSQ with active components Q_1' and $Q_{\{1,2\}}'$, which has the same second (refined) partition $P_{\{1,2\}}$ as $\mathbf{Q}$, but its first partition consists of two nonconvex cells which are unions of an interval and a half-line in the following way:

$$P_1' = \{(-\infty, t_1) \cup [t_2, t_3), [t_1, t_2) \cup [t_3, \infty)\}$$

and $\mathbf{Q}'$ also satisfies the centroid condition. Hence $D_{\{1,2\}}(\mathbf{Q}') = D_{\{1,2\}}(\mathbf{Q})$. Define $\mathbf{Q}^{.1h}$ and $\mathbf{Q}'^{.1h}$ as instances of $\mathbf{Q}$ and $\mathbf{Q}'$, respectively, when $t_2 - 6 = 2\epsilon_2 = 0.1h$ (and the centroid condition is satisfied).

For a given t_2, y_1 and y_2 are determined by the centroid condition:

$$\begin{aligned} y_1 &= \frac{2h(2-3h) + 2h(4-h) + 3\cdot 2\epsilon_2(6+\epsilon_2)}{2h+2h+3\cdot 2\epsilon_2} \\ &= 3 - 2h + 3\epsilon_2\left(1 + \frac{3-2\epsilon_2}{2h+3\epsilon_2}\right) \end{aligned}$$

$$\begin{aligned} y_2 &= \frac{3\cdot 2(h-\epsilon_2)(6+h+\epsilon_2) + 3\cdot 2h(14+3h)}{3\cdot 2(h-\epsilon_2) + 3\cdot 2h} \\ &= 10 + 2h + \epsilon_2\left(1 + \frac{4}{2h-\epsilon_2}\right). \end{aligned}$$

These imply that

$$y_1 + y_2 = 13 + \epsilon_2\left(4 + 3\frac{3-2\epsilon_2}{2h+3\epsilon_2} + \frac{4}{2h-\epsilon_2}\right) > 2(6+0.1h).$$

So moving the boundary from t_2 into $t_2^{.1h} = 6+0.1h$ and applying Lemma 4 case ii) with SQ Q_1, interval $[t_2, 6+0.1h]$, and code points (y_2, y_1), we get that $D_1(\mathbf{Q}) \ge D_1(\mathbf{Q}^{.1h})$.

On the other hand, for a given t_2, also y_1' and y_2' code points of Q_1' are determined by the centroid condition:

$$\begin{aligned} y_1' &= \frac{2h(2-3h) + 3\cdot 2(h-\epsilon_2)(6+h+\epsilon_2)}{2h + 3\cdot 2(h-\epsilon_2)} \\ &= 5 - 3\epsilon_2\frac{1+\epsilon_2}{4h-3\epsilon_2} \end{aligned}$$

$$\begin{aligned} y_2' &= \frac{2h(4-h) + 3\cdot 2\epsilon_2(6+\epsilon_2) + 3\cdot 2h(14+3h)}{2h + 3\cdot 2\epsilon_2 + 3\cdot 2h} \\ &= 11.5 + 2h - \frac{\epsilon_2}{2}\left(3 + \frac{33-15\epsilon_2}{4h+3\epsilon_2}\right) \end{aligned}$$

which, for $\mathbf{Q}'^{.1h}$ (substituting $\epsilon_2 = h/20$), are in turn

$$y_1'^{.1h} = 5 - \frac{3+0.15h}{77} \quad , \quad y_2'^{.1h} = 11.5 - \frac{33}{166} + 2h - \frac{117h}{1660}.$$

These imply that

$$\begin{aligned} y_1'^{.1h} + y_2'^{.1h} &= 16.5 - \frac{3}{77} - \frac{33}{166} + 2h - \frac{h}{20}\left(\frac{3}{77} + \frac{117}{83}\right) \\ &> 2(6+0.1h). \end{aligned}$$

So moving the boundary from $t_2^{1h} = 6+0.1h$ back into $t_2 = 6+2\epsilon_2$ and applying Lemma 4 case ii) with SQ $Q_1'^{.1h}$, interval $[t_2, 6+0.1h]$, and code points (y_2, y_1), we get that $D_1(\mathbf{Q}'^{.1h}) \ge D_1(\mathbf{Q}')$.

Now, we compare the distortions $\bar{D}(\mathbf{Q})$ and $\bar{D}(\mathbf{Q}')$. Recalling $D_{\{1,2\}}(\mathbf{Q}') = D_{\{1,2\}}(\mathbf{Q})$,

$$\begin{aligned} &\bar{D}(\mathbf{Q}) - \bar{D}(\mathbf{Q}') \\ &= \omega D_1(\mathbf{Q}) + D_{\{1,2\}}(\mathbf{Q}) - \omega D_1(\mathbf{Q}') - D_{\{1,2\}}(\mathbf{Q}') \\ &= \omega(D_1(\mathbf{Q}) - D_1(\mathbf{Q}')). \end{aligned}$$

By the inequalities above, the difference factor can be lower bounded by $D_1(\mathbf{Q}^{.1h}) - D_1(\mathbf{Q}'^{.1h})$. By definition,

$$\begin{aligned} &16h(D_1(\mathbf{Q}^{.1h}) - D_1(\mathbf{Q}'^{.1h})) \\ &= \int_{2-4h}^{2-2h} (x - y_1^{.1h})^2\,dx - \int_{2-4h}^{2-2h} (x - y_1'^{.1h})^2\,dx \\ &\quad + \int_{4-2h}^{4} (x - y_1^{.1h})^2\,dx - \int_{4-2h}^{4} (x - y_2'^{.1h})^2\,dx \\ &\quad + \int_{6}^{6+0.1h} 3(x - y_1^{.1h})^2\,dx - \int_{6}^{6+0.1h} 3(x - y_2'^{.1h})^2\,dx \\ &\quad + \int_{6+0.1h}^{6+2h} 3(x - y_2^{.1h})^2\,dx - \int_{6+0.1h}^{6+2h} 3(x - y_1'^{.1h})^2\,dx \\ &\quad + \int_{14+2h}^{14+4h} 3(x - y_2^{.1h})^2\,dx - \int_{14+2h}^{14+4h} 3(x - y_2'^{.1h})^2\,dx \end{aligned}$$

where

$$y_1^{.1h} = \frac{138}{43} - \frac{1597h}{860} \quad , \quad y_2^{.1h} = \frac{394}{39} + 2.05h$$

are the values of y_1, y_2 for $\mathbf{Q}^{.1h}$ (substituting $\epsilon_2 = h/20$), respectively. We use the following elementary identity

$$\int_a^b (x-y)^2\,dx - \int_a^b (x-y')^2\,dx = (b-a)(y'-y)(b+a-(y+y'))$$

to see that the above sum is

$$\begin{aligned} &2h(y_1'^{.1h} - y_1^{.1h})(4 - 6h - (y_1^{.1h} + y_1'^{.1h})) \\ &+ 2h(y_2'^{.1h} - y_1^{.1h})(8 - 2h - (y_1^{.1h} + y_2'^{.1h})) \\ &+ 0.3h(y_2'^{.1h} - y_1^{.1h})(12 + 0.1h - (y_1^{.1h} + y_2'^{.1h})) \\ &+ 5.7h(y_1'^{.1h} - y_2^{.1h})(12 + 2.1h - (y_2^{.1h} + y_1'^{.1h})) \\ &+ 6h(y_2'^{.1h} - y_2^{.1h})(28 + 6h - (y_2^{.1h} + y_2'^{.1h})). \end{aligned}$$

After substitution, calculation, and putting together, we get that

$$\bar{D}(\mathbf{Q}) - \bar{D}(\mathbf{Q}') \geq \omega(0.697 - 4.472h - 2.07h^2).$$

Re-scaled by the factor $\omega_{\{1,2\}}$, this gives the statement of the lemma. □

Proof of Theorem 1: This follows obviously from Lemma 5 since there the right side is positive if $\omega_{\{1\}} > 0$ and $h < 1/7$. □

Proof of Lemma 1: Let $\{C_1, \ldots, C_N\}$ be the partition of Q. Note that the support of f_ϵ is contained in $[b_1 - \epsilon, b_K + w_K + \epsilon] \subseteq [0, 1]$. Now

$$\begin{aligned}
&|D(\mu_\epsilon, Q) - D(\mu, Q)| \\
&= \left| \sum_{i=1}^{N} \int_{C_i} (x - y_i)^2 f_\epsilon(x)\, dx - \sum_{i=1}^{N} \int_{C_i} (x - y_i)^2 f(x)\, dx \right| \\
&= \left| \sum_{i=1}^{N} \int_{C_i} (x - y_i)^2 (f_\epsilon(x) - f(x))\, dx \right| \\
&\leq \sum_{i=1}^{N} \int_{C_i \cap [0,1]} (x - y_i)^2 |f_\epsilon(x) - f(x)|\, dx.
\end{aligned}$$

Here x, $y_i \in [0, 1]$ implies $(x - y_i)^2 \leq 1$, thus

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{C_i \cap [0,1]} (x - y_i)^2 |f_\epsilon(x) - f(x)|\, dx \\
&\quad \leq \sum_{i=1}^{N} \int_{C_i} |f_\epsilon(x) - f(x)|\, dx = \int_{\mathcal{R}} |f_\epsilon(x) - f(x)|\, dx.
\end{aligned}$$

Since $|f_\epsilon(x) - f(x)|$ can be written as

$$\begin{aligned}
&\sum_{i=1}^{K} \frac{a_i}{2\epsilon} \big((x - b_i + \epsilon) \mathbb{I}_{\{x \in [b_i - \epsilon, b_i]\}} \\
&\quad + (b_i + \epsilon - x) \mathbb{I}_{\{x \in [b_i, b_i + \epsilon]\}} \\
&\quad + (x - b_i - w_i + \epsilon) \mathbb{I}_{\{x \in [b_i + w_i - \epsilon, b_i + w_i]\}} \\
&\quad + (b_i + w_i + \epsilon - x) \mathbb{I}_{\{x \in [b_i + w_i, b_i + w_i + \epsilon]\}} \big)
\end{aligned}$$

and the intervals above are disjoint, its integral is

$$\begin{aligned}
&\int_{\mathcal{R}} |f_\epsilon(x) - f(x)|\, dx \\
&= \sum_{i=1}^{K} \frac{a_i}{2\epsilon} \left(\int_{b_i - \epsilon}^{b_i} x - b_i + \epsilon\, dx + \int_{b_i}^{b_i + \epsilon} b_i + \epsilon - x\, dx \right. \\
&\quad \left. + \int_{b_i + w_i - \epsilon}^{b_i + w_i} x - b_i - w_i + \epsilon\, dx + \int_{b_i + w_i}^{b_i + w_i + \epsilon} b_i + w_i + \epsilon - x\, dx \right) \\
&= \sum_{i=1}^{K} \frac{a_i}{2\epsilon} 4 \int_{0}^{\epsilon} x\, dx = \epsilon \sum_{i=1}^{K} a_i
\end{aligned}$$

giving the desired bound. □

Now we can prove Theorem 2 approximating the pdf in Theorem 1 by a continuous one. Consider f and μ defined by (7). Let f_ϵ and μ_ϵ be their continuous approximation, as is (5) for (3), supported on the four disjoint intervals $[2 - 4h - \epsilon, 2 - 2h + \epsilon]$, $[4 - 2h - \epsilon, 4 + \epsilon]$, $[6 - \epsilon, 6 + 2h + \epsilon]$, and $[14 + 2h - \epsilon, 14 + 4h + \epsilon]$. The following lemma gives a more technical version of Theorem 2:

Lemma 7: If $\omega \leq 1/300$, $h \leq 3/16$, and $\epsilon < \min(1 - 4h, h)$ then for the source μ_ϵ corresponding to f_ϵ above and for any convex 2-RSQ $\mathbf{Q} \in \mathcal{Q}_{1,1}$

$$\begin{aligned}
&\bar{D}(\mathbf{Q}) - \inf_{\mathbf{Q}' \in \mathcal{Q}_{1,1}} \bar{D}(\mathbf{Q}') \\
&\geq \omega_{\{1\}}(0.697 - 4.472h - 2.07h^2) - 14(\omega_{\{1\}} + \omega_{\{1,2\}}) \frac{\epsilon}{h}.
\end{aligned}$$

Proof of Lemma 7: As in the proof of Lemma 5, there is an optimal MRSQ for μ_ϵ among all convex MRSQs with given rates. So we can assume w.l.o.g. that $\mathbf{Q}$ is optimal for μ_ϵ and $\omega_{\{1\}}$, $\omega_{\{1,2\}}$ among all convex 2-RSQs in $\mathcal{Q}_{1,1}$. Then it has to satisfy the centroid condition for μ_ϵ, that is, all its code points lie in the convex hull of the support of μ_ϵ, which is $[2 - 4h - \epsilon, 14 + 4h + \epsilon] \subset [1, 15]$ (from $\epsilon < 1 - 4h$). Thus applying Lemma 1 re-scaled for the interval $[1, 15]$ for f, μ, f_ϵ, μ_ϵ above and Q_1, $Q_{\{1,2\}}$ gives

$$\begin{aligned}
&\max(|D(\mu_\epsilon, Q_1) - D(\mu, Q_1)|, |D(\mu_\epsilon, Q_{\{1,2\}}) - D(\mu, Q_{\{1,2\}})|) \\
&\qquad \leq 14\epsilon \frac{1 + 1 + 3 + 3}{16h} = 7\frac{\epsilon}{h}
\end{aligned}$$

whenever $\epsilon < \min(1 - 4h, h)$. Hence, using (2),

$$\begin{aligned}
&\bar{D}(\mu, \mathbf{Q}) - \bar{D}(\mu_\epsilon, \mathbf{Q}) \\
&= \omega_{\{1\}}(D(\mu, Q_1) - D(\mu_\epsilon, Q_1)) \\
&\quad + \omega_{\{1,2\}}(D(\mu, Q_{\{1,2\}}) - D(\mu_\epsilon, Q_{\{1,2\}})) \\
&\leq 7(\omega_{\{1\}} + \omega_{\{1,2\}}) \frac{\epsilon}{h}.
\end{aligned} \tag{9}$$

Now, since $\mathbf{Q}$ is convex, according to Lemma 5, for any $\mathbf{Q}' \in \mathcal{Q}_{1,1}$ optimal for μ, we have

$$\bar{D}(\mu, \mathbf{Q}) - \bar{D}(\mu, \mathbf{Q}') \geq \omega_{\{1\}}(0.697 - 4.472h - 2.07h^2). \tag{10}$$

$\mathbf{Q}'$ has to satisfy the centroid condition for μ, that is, its code points are in the convex hull of the support of μ, which is $[2 - 4h, 14 + 4h] \subset [1, 15]$ (from $4h \leq 3/4$). Thus applying now Lemma 1 as above, but for the active components Q'_1, $Q'_{\{1,2\}}$ of $\mathbf{Q}'$ gives

$$\begin{aligned}
&\max(|D(\mu_\epsilon, Q'_1) - D(\mu, Q'_1)|, |D(\mu_\epsilon, Q'_{\{1,2\}}) - D(\mu, Q'_{\{1,2\}})|) \\
&\qquad \leq 7\frac{\epsilon}{h}
\end{aligned}$$

whenever $\epsilon < \min(1 - 4h, h)$, which implies

$$\begin{aligned}
&\bar{D}(\mu_\epsilon, \mathbf{Q}') - \bar{D}(\mu, \mathbf{Q}') \\
&= \omega_{\{1\}}(D(\mu_\epsilon, Q'_1) - D(\mu, Q'_1)) \\
&\quad + \omega_{\{1,2\}}(D(\mu_\epsilon, Q'_{\{1,2\}}) - D(\mu, Q'_{\{1,2\}})) \\
&\leq 7(\omega_{\{1\}} + \omega_{\{1,2\}}) \frac{\epsilon}{h}.
\end{aligned} \tag{11}$$

Putting together (9), (10), and (11), we get that

$$\begin{aligned}
&\bar{D}(\mu_\epsilon, \mathbf{Q}) - \bar{D}(\mu_\epsilon, \mathbf{Q}') \\
&\geq \omega_{\{1\}}(0.697 - 4.472h - 2.07h^2) - 14(\omega_{\{1\}} + \omega_{\{1,2\}}) \frac{\epsilon}{h}.
\end{aligned}$$

This gives the statement of the lemma. □

Proof of Theorem 2: This follows obviously from Lemma 7 since there the right side is positive if $\omega_{\{1\}} > 0$, $h < 1/7$, and

$$\epsilon < \frac{\omega}{\omega+1}\frac{h}{14}(0.697 - 4.472h - 2.07h^2).$$

For example, $h = 1/14$, $\omega = 1/300$, and $\epsilon = 6 \cdot 10^{-6}$ will do. □

Remark 1: The counterexamples in the results above are for $R_1 = R_2 = 1$. However, it seems possible to extend these to higher rates R_1, R_2. In particular, for any rates and levels $N_1 = 2^{R_1}$, $N_2 = 2^{R_2}\,(\geq 2)$, let $N \stackrel{\text{def}}{=} N_1 N_2$. Then for the discrete distribution with probability mass function

$$\{p, p, p, \ldots, (1-(N-2)p)/2, (1-(N-2)p)/2\}$$

on alphabet $\{1, 2, 3, \ldots, N-1, N+W\}$, when $W \gg 1$ and p and $\omega = \omega_{\{1\}}/\omega_{\{1,2\}}$ is small enough, a similar argument as for Proposition 1 must show that an optimal 2-RSQ cannot be convex. Moreover, it must be possible to extend this for absolutely continuous sources with bounded (continuous) densities replacing the atoms by intervals of a pdf with length $h \ll 1$ as in (7) (and then (5)). Giving an accurate proof for this generalization is far beyond the scope of this paper.

Remark 2: Also the results above are stated for the case of 2-RSQs. Again, as noted also in [6], it seems plausible to extend these to more than 2 resolution levels. In particular, if we have 3 (or more) resolution levels with given rates $R_1 = R_2 = 1$, R_3,... and corresponding weights $\omega_{\{1\}}$, $\omega_{\{1,2\}}$, $\omega_{\{1,2,3\}}, \ldots > 0$ such that $\omega_{\{1\}}$ and $\omega_{\{1,2\}}$ still satisfy $\omega \ll 1$ and the other weights are sufficiently small compared to them, and we keep the distributions defined by f in (7) (and f_ϵ), then an optimal 3-RSQ (M-RSQ) must have the same active components Q_1 (that is nonconvex) and $Q_{\{1,2\}}$ as in the 2-RSQ case above and components $Q_{\{1,2,3\}}$,... dividing further each of the four support intervals in (7) uniformly. Thus our theorems must extend to M-RSQs for $M > 2$. Giving a detailed proof for this is beyond the scope of this paper, as well.

IV. Conclusions and Future Work

It has been proven earlier that codecell convexity may preclude optimality for MRSQs showing a discrete counterexample. Here we have proven that convexity of FR MRSQs may preclude optimality for the MSE fidelity criterion also for absolutely continuous sources with bounded continuous pdfs over bounded regions.

Our counterexample are given for a $\{1, 1\}$ rate 2-RSQ setup with certain weights in the total distortion. We have given directions in Remarks 1,2 for extension to higher rates and more resolutions, but left the detailed calculations for future work. It still remains possible that under some practical (perhaps combined) constraints on the weights, the rates, and/or the distribution, there are always convex optimal MRSQs (as in the case of the uniform source). See [2, Section VII] for related conjectures and statements, where both conditions on the weight ratios and high-rate assumptions are considered.

We have not touched issues such as the other types of extensions of SQ model mentioned in Section I-B. It remains an open question how the results can be generalized for fidelity criterions other than MSE.

Proposition 1 can be obviously generalized to vector quantizers using the fact that the intersection of a convex cell and a straight line is convex. Similar generalizations of Theorems 1 and 2 are not so obvious (since the support of a multidimensional pdf cannot be on a line), however, we think that they can be done by some approximation method without actual difficulty.

Another open question is whether the results can be extended to VR or EC quantization, for example, based on the Lagrangian formulation used in [20]. However, as stated there, very little is known concerning VR quantizers achieving minimum distortion; nor is it known whether an optimal VR quantizer always exists. Even in the Lagrangian sense, the analogous to the nearest neighbor condition of Section I-A is much more complicated.

For EC quantization, the existence of an optimal ECSQ is known for several sources [15], [21]. In [6], it is shown that also for EC MRSQs (and so MDSQs), codecell convexity may preclude optimality for some discrete sources, that is, Proposition 1 can be extended to this case. Whether the analogous extensions of Theorems 1 and 2 hold is especially interesting in cases of finite-level EC MDSQs/MRSQs, because for sources with pdfs, as mentioned and stated in Section II, whereas on one hand convexity of FR MDSQs [2] and MRSQs (Theorem 1) may preclude optimality, on the other hand for finite-level ECSQs the optimal distortion can be arbitrarily approximated by the convex ones [15], [16].

We are currently investigating such models.

Appendix
Proofs of the Auxiliary Lemmata

Proof of Lemma 2: The centroid condition implies that the optimal code point is the conditional mean of the source in the cell:

$$\frac{2w_1a_1(-1-w_1) + 2w_2a_2(1+w_2)}{2w_1a_1 + 2w_2a_2} = \frac{a_2w_2(1+w_2) - a_1w_1(1+w_1)}{a_1w_1 + a_2w_2}.$$

□

Proof of Lemma 3: By Lemma 2, the optimal code point is

$$c = \frac{a_2w_2(1+w_2h) - a_1w_1(1+w_1h)}{a_1w_1 + a_2w_2}$$

and the distortion contribution is

$$\begin{aligned} &d_{h,w_1,a_1,w_2,a_2} \\ &= \int_{-1-2w_1h}^{-1} \frac{a_1}{h}(x-c)^2\,dx + \int_{1}^{1+2w_2h} \frac{a_2}{h}(x-c)^2\,dx \\ &= \frac{a_1}{h}\int_{-1-2w_1h}^{-1} x^2 - 2xc + c^2\,dx + \frac{a_2}{h}\int_{1}^{1+2w_2h} x^2 - 2xc + c^2\,dx \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\bigg(\frac{1}{3}(a_1((1+2w_1h)^3-1)+a_2((1+2w_2h)^3-1))\\
&\quad - a_1c(1-(1+2w_1h)^2)-a_2c((1+2w_2h)^2-1)\\
&\quad + 2a_1c^2w_1h+2a_2c^2w_2h\bigg)\\
&= 2(a_1w_1(1+2w_1h+4w_1^2h^2/3)\\
&\quad + a_2w_2(1+2w_2h+4w_2^2h^2/3)\\
&\quad - 2c(a_2w_2(1+w_2h)-a_1w_1(1+w_1h))\\
&\quad + c^2(a_1w_1+a_2w_2)).
\end{aligned}
$$

Now substituting c, the last two terms in the outermost parentheses together give

$$-\frac{(a_2w_2(1+w_2h)-a_1w_1(1+w_1h))^2}{a_1w_1+a_2w_2}$$

and so

$$
\begin{aligned}
&\frac{3}{2}(a_1w_1+a_2w_2)d_{h,w_1,a_1,w_2,a_2}\\
&= (a_1w_1+a_2w_2)\cdot\big(a_1w_1(3+6w_1h+4w_1^2h^2)\\
&\quad + a_2w_2(3+6w_2h+4w_2^2h^2)\big)\\
&\quad - 3(a_2w_2(1+w_2h)-a_1w_1(1+w_1h))^2\\
&= (a_1w_1+a_2w_2)\cdot(3(a_1w_1+a_2w_2)\\
&\quad + 6h(a_1w_1^2+a_2w_2^2)+4h^2(a_1w_1^3+a_2w_2^3))\\
&\quad - 3(a_2w_2-a_1w_1+(a_2w_2^2-a_1w_1^2)h)^2\\
&= 3(a_1w_1+a_2w_2)^2+6h(a_1w_1+a_2w_2)(a_1w_1^2+a_2w_2^2)\\
&\quad + 4h^2(a_1w_1+a_2w_2)(a_1w_1^3+a_2w_2^3)\\
&\quad - 3(a_2w_2-a_1w_1)^2-6h(a_2w_2-a_1w_1)(a_2w_2^2-a_1w_1^2)\\
&\quad - 3(a_2w_2^2-a_1w_1^2)^2h^2\\
&= 12a_1a_2w_1w_2+12ha_1a_2w_1w_2(w_1+w_2)+h^2\cdot\\
&\quad (a_1^2w_1^4+a_2^2w_2^4+4a_1a_2w_1w_2(w_1^2+w_2^2)+6a_1a_2w_1^2w_2^2)\\
&= a_1a_2w_1w_2(12+6s)\\
&\quad + h^2((a_2w_2^2-a_1w_1^2)^2+4a_1a_2w_1w_2(w_1+w_2)^2)\\
&= a_1a_2w_1w_2(12+6s+s^2)+h^2(a_2w_2^2-a_1w_1^2)^2
\end{aligned}
$$

giving the first equation. The next two equations follow from trivial substitution. The last statement is obvious. □

Proof of Lemma 4: For any $x\in[a,b]$, in case i), $2x\ge 2a>y+\tilde{y}$, that is, $x-y>\tilde{y}-x$, and also $x-y>x-\tilde{y}$, implying together $|x-y|>|x-\tilde{y}|$. In case ii), $y+\tilde{y}>2b\ge 2x$, that is, $y-x>x-\tilde{y}$, and also $y-x>\tilde{y}-x$, implying together $|x-y|>|x-\tilde{y}|$ again. Hence $(x-y)^2-(x-\tilde{y})^2>0$ on $x\in[a,b]$ anyway. Using (1) and the definition of Q', we have

$$D(Q)-D(Q')=\int_a^b (x-y)^2-(x-\tilde{y})^2\,\mu(dx)>0$$

proving the first statement. The second statement is obvious. Also the third one that follows from the definition of the centroid condition. □

Proof of Lemma 6: Assume that for some 4-level quantizer Q, $D(Q)<D(Q^0_{\{1,2\}})$. We can assume w.l.o.g. that each boundary point of Q is in some (closed) interval of the support of (7) since this can be reached without changing $D(Q)$. If a cell of Q contains one of the gaps $[2-2h,4-2h]$ or $[6+2h,14+2h]$ of the support then it is easy to see that the optimal distortion contribution from this cell does not increase if we dissolve this gap moving the support interval $[2-4h,2-2h]$ or $[14+2h,14+4]$ beside $[4-2h,4]$ or $[6,6+2h]$, respectively. Thus $D(Q)$ is lower bounded by the optimal distortion D' for the density $(\mathbb{I}_{\{x\in[4-4h,4]\}}+3\mathbb{I}_{\{x\in[6,6+4h]\}})/(16h)$, which is reached by some SQ, say Q' with boundary points $\{t_i\}_{i=1}^3$. We can still assume that each t_i is in one of these two united support intervals, namely k of them are in $[4-4h,4]$ and $3-k$ of them are in $[6,6+4h]$. Then the total distortion from the left-most k cells in $[4-4h,4]$ is (with $t_0\stackrel{\text{def}}{=}4-4h$)

$$
\begin{aligned}
\sum_{i=1}^k d_{h,\frac{t_i-t_{i-1}}{2h},1/16,0,0} &= \frac{1}{192h}\sum_{i=1}^k (t_i-t_{i-1})^3\\
&\ge \frac{k}{192h}\left(\frac{1}{k}\sum_{i=1}^k (t_i-t_{i-1})\right)^3\\
&= \frac{(4h+t_k-4)^3}{192hk^2}\\
&= \frac{h^2}{3}\frac{(1-x_1)^3}{k^2}\quad \text{with } x_1\stackrel{\text{def}}{=}\tfrac{4-t_k}{4h}
\end{aligned}
$$

if $k>0$ and 0 if $k=0$ (then $x_1\stackrel{\text{def}}{=}1$). Similarly, the total distortion from the right-most $3-k$ cells in $[6,6+4h]$ is at least $h^2\frac{(1-x_2)^3}{(3-k)^2}$ with $x_2\stackrel{\text{def}}{=}\frac{t_{k+1}-6}{4h}$ if $k<3$ and 0 if $k=3$ (then $x_2\stackrel{\text{def}}{=}1$). Note that $x_1,x_2\in[0,1]$. Moreover, the distortion of Q' from the cell $[t_k,t_{k+1}]$ is $d_{h,2x_1,1/16,2x_2,3/16}$ if $x_1+x_2>0$ (and 0 if $x_1=x_2=0$), and thus

$$D'\ge\frac{h^2}{3}\left(\frac{(1-x_1)^3}{k^2}+\frac{3(1-x_2)^3}{(3-k)^2}\right)+d_{h,2x_1,1/16,2x_2,3/16}. \tag{12}$$

We show that in all cases, $D(Q)\ge D'\ge D(Q^0_{\{1,2\}})$, that contradicts to our assumption.

Case $k=0$ ($x_1=1$): Then $D'\ge d_{h,2,1/16,2x_2,3/16}$ from (12), which, using Lemma 3, is at least $d_{h,2,1/16,0,3/16}=h^2/3=D(Q^0_{\{1,2\}})$.

Case $k=3$ ($x_2=1$): Then, similarly, $D'\ge d_{h,2x_1,1/16,2,3/16}\ge d_{h,0,1/16,2,3/16}=h^2>D(Q^0_{\{1,2\}})$ again.

Case $k=1$: Then for $x_1=x_2=0$, (12) is obviously $(1+3/4)h^2/3>D(Q^0_{\{1,2\}})$, otherwise it is

$$
\begin{aligned}
D' &\ge \frac{h^2}{3}\left((1-x_1)^3+\frac{3(1-x_2)^3}{4}\right)+d_{h,2x_1,1/16,2x_2,3/16}\\
&= \frac{h^2}{3}+\frac{h^2}{3}\frac{A_1}{4(x_1+3x_2)}
\end{aligned}
$$

where A_1 is

$$
\begin{aligned}
&4(x_1+3x_2)\\
&\quad\cdot\left((1-x_1)^3+\frac{3(1-x_2)^3}{4}-1+\frac{3d_{h,2x_1,1/16,2x_2,3/16}}{h^2}\right)\\
&= (x_1+3x_2)(4(1-x_1)^3+3(1-x_2)^3-4)\\
&\quad + 4(3x_1x_2(3/h^2+6(x_1+x_2)/h+4(x_1+x_2)^2)\\
&\qquad + (3x_2^2-x_1^2)^2)\\
&\ge (x_1+3x_2)\\
&\quad\cdot(4(1-3x_1+3x_1^2-x_1^3)+3(1-3x_2+3x_2^2-x_2^3)-4)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4x_1x_2(256 + 96(x_1 + x_2) + 12(x_1 + x_2)^2) \\
&+ 4(3x_2^2 - x_1^2)^2 \\
&\text{(from } h \le 3/16) \\
= {}& 3x_1(1 - 2x_1)^2 + 9x_2(1/4 + 3(x_2 - 1/2)^2) + 979x_1x_2 \\
&+ 420x_1^2x_2 + 393x_1x_2^2 \\
&+ 45x_1x_2^3 + 72x_1^2x_2^2 + 36x_1^3x_2 + 27x_2^4 \\
\ge {}& 0
\end{aligned}
$$

(each term is nonnegative). So $D' \ge h^2/3 = D(Q^0_{\{1,2\}})$ again.

Case $k = 2$: Then, similarly, for $x_1 = x_2 = 0$, (12) is $(1/4 + 3)h^2/3 > D(Q^0_{\{1,2\}})$, otherwise it is

$$
\begin{aligned}
D' &\ge \frac{h^2}{3}\left(\frac{(1 - x_1)^3}{4} + 3(1 - x_2)^3\right) + d_{h,2x_1,1/16,2x_2,3/16} \\
&= \frac{h^2}{3} + \frac{h^2}{3}\frac{A_2}{4(x_1 + 3x_2)},
\end{aligned}
$$

where A_2 is

$$
\begin{aligned}
&4(x_1 + 3x_2) \\
&\quad \cdot \left(\frac{(1 - x_1)^3}{4} + 3(1 - x_2)^3 - 1 + \frac{3d_{h,2x_1,1/16,2x_2,3/16}}{h^2}\right) \\
= {}& (x_1 + 3x_2)((1 - x_1)^3 + 12(1 - x_2)^3 - 4) \\
&+ 4(3x_1x_2(3/h^2 + 6(x_1 + x_2)/h + 4(x_1 + x_2)^2) \\
&\quad + (3x_2^2 - x_1^2)^2) \\
\ge {}& (x_1 + 3x_2) \\
&\cdot (1 - 3x_1 + 3x_1^2 - x_1^3 + 12(1 - 3x_2 + 3x_2^2 - x_2^3) - 4) \\
&+ 4x_1x_2(256 + 96(x_1 + x_2) + 12(x_1 + x_2)^2) \\
&+ 4(3x_2^2 - x_1^2)^2 \\
= {}& 3x_1(3 - x_1) + 27x_2(1 - 2x_2)^2 + 979x_1x_2 \\
&+ 3x_1^3 + 393x_1^2x_2 + 420x_1x_2^2 \\
&+ 3x_1^4 + 45x_1^3x_2 + 72x_1^2x_2^2 + 36x_1x_2^3 \\
\ge {}& 0
\end{aligned}
$$

again. So $D' \ge h^2/3 = D(Q^0_{\{1,2\}})$. □

Proof of $D_1(\mathbf{Q}^0) \le D_1(\mathbf{Q})$ in Lemma 5, Case $t_2 < 6$: Consider three subcases:

Case $t_2 \in [4, 6)$: Then $D_1(\mathbf{Q}) = D_1(\mathbf{Q}^0)$ for $\mu([4, 6]) = 0$.

Case $t_2 \in [4 - 2h, 4)$: Then $y_1 \ge 2 - 3h$ and $y_2 \ge y_2^*$, where

$$
y_2^* = \frac{(4 - h)/8 + 3(6 + h)/8 + 3(14 + 3h)/8}{7/8} = \frac{64 + 11h}{7}
$$

is the optimal code point in a cell consisting of the interval $[4 - 2h, 14 + 4h]$. These and $h \le 3/16$ imply that

$$
y_1 + y_2 \ge \frac{78 - 10h}{7} \ge \frac{87}{8} > 8.
$$

Moving the boundary from t_2 into 4 and applying Lemma 4 case ii) with SQ Q_1, interval $[t_2, 4]$, and code points (y_1, y_2), we get that $D_1(\mathbf{Q}) \ge D(Q'')$ where the SQ Q'' has its only boundary at 4 and satisfies the centroid condition, and so again $D(Q'') = D_1(\mathbf{Q}^0)$.

Case $t_2 < 4 - 2h$: Then $D_1(\mathbf{Q})$ is lower bounded by the distortion contribution from the cell consisting of $[4 - 2h, 14 + 4h]$ with code point y_2^* above. The latter can written as

$$
\begin{aligned}
&\frac{1}{16h}\cdot \\
&\left[\int_{4-2h}^{4}(x - y_2^*)^2dx + \int_{6}^{6+2h}3(x - y_2^*)^2dx + \int_{14+2h}^{14+4h}3(x - y_2^*)^2dx\right] \\
= {}& \frac{1}{16h}\left[\int_{-h}^{h}x^2\,dx + 3\int_{-h}^{h}x^2\,dx + 3\int_{-h}^{h}x^2\,dx\right] + \\
&\left((4 - h - y_2^*)^2 + 3(6 + h - y_2^*)^2 + 3(14 + 3h - y_2^*)^2\right)/8 \\
= {}& \Big[7h^2/3 + 16 + h^2 - 8h - 2(4 - h)y_2^* + {y_2^*}^2 \\
&+ 108 + 3h^2 + 36h - 6(6 + h)y_2^* + 3{y_2^*}^2 \\
&+ 588 + 27h^2 + 252h - 6(14 + 3h)y_2^* + 3{y_2^*}^2\Big]/8 \\
= {}& \left[712 + 280h + 100h^2/3 - 2(64 + 11h)y_2^* + 7{y_2^*}^2\right]/8 \\
= {}& \left[4984 + 1960h + 700h^2/3 - (64 + 11h)^2\right]/56 \\
= {}& \left(111 + 69h + 337h^2/24\right)/7,
\end{aligned}
$$

that is (term-wise) greater than $\frac{49}{4} + 6.5h + \frac{4h^2}{3} = D_1(\mathbf{Q}^0)$. □

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REFERENCES

[1] D. Pollard, "Quantization and the method of k-means," *IEEE Transactions on Information Theory*, vol. IT-28, no. 2, pp. 199–205, Mar. 1982.
[2] S. Dumitrescu and X. Wu, "On properties of locally optimal multiple description scalar quantizers with convex cells," *IEEE Transactions on Information Theory*, vol. IT-55, no. 12, Dec. 2009.
[3] J. K. Wolf, A. D. Wyner, and J. Ziv, "Source coding for multiple descriptions," *Bell Sys. Tech. Jour.*, vol. 59, pp. 1417–1426, 1980.
[4] M. Effros, "Optimal multiple description and multiresolution scalar quantizer design," in *Information Theory and Applications Workshop*. Piscataway, NJ: IEEE Comput. Soc. Press, Aug. 2008, pp. 70–76, (San Diego, CA, 27 Jan–1 Feb, 2008).
[5] S. Dumitrescu and X. Wu, "Algorithms for optimal multi-resolution quantization," *Journal of Algorithms*, vol. 50, pp. 1–22, Jan. 2004.
[6] D. Muresan and M. Effros, "Quantization as histogram segmentation: optimal scalar quantizer design in network systems," *IEEE Transactions on Information Theory*, vol. IT-54, no. 1, p. 344366, Jan. 2008.
[7] V. A. Vaishampayan, "Design of multiple description scalar quantizers," *IEEE Transactions on Information Theory*, vol. IT-39, no. 3, pp. 821–834, May 1993.
[8] M. Effros and L. Schulman, "Rapid near-optimal vq design with a deterministic data net," in *Proceedings 2004 IEEE International Symposium on Information Theory*, Jun. 2004, p. 299, (Chicago, IL, June 27-July 2, 2004.).
[9] H. Brunk and N. Farvardin, "Fixed-rate successively refinable scalar quantizers," in *Proceedings of the IEEE Data Compression Conference, DCC'96*. IEEE Comput. Soc. Press, Apr. 1996, pp. 250–259, (Snowbird, UT, March 1996).
[10] D. Muresan and M. Effros, "Quantization as histogram segmentation: globally optimal scalar quantizer design in network systems," in *Proceedings of the IEEE Data Compression Conference, DCC 2002*. Snowbird, UT: IEEE Comput. Soc. Press, Apr. 2002, pp. 302–311.

[11] X. Wu and S. Dumitrescu, "On optimal multi-resolution scalar quantization," in *Proceedings of the IEEE Data Compression Conference, DCC 2002*. Snowbird, UT: IEEE Comput. Soc. Press, Apr. 2002, pp. 322–331.

[12] S. Dumitrescu and X. Wu, "Lagrangian optimization of two-description scalar quantizers," *IEEE Transactions on Information Theory*, vol. IT-53, no. 11, pp. 3990–4012, Nov. 2007.

[13] S. Dumitrescu and X. Wu, "Optimal two-description scalar quantizer design," *Algorithmica*, vol. 41, no. 4, pp. 269–287, Apr. 2005.

[14] M. Effros and D. Muresan, "Codecell contiguity in optimal fixed-rate and entropy-constrained network scalar quantization," in *Proceedings of the Data Compression Conference, DCC 2002*. Snowbird, UT: IEEE Comput. Soc. Press, Apr. 2002, pp. 312–321.

[15] A. György and T. Linder, "On the structure of optimal entropy-constrained scalar quantizers," *IEEE Transactions on Information Theory*, vol. IT-48, no. 2, pp. 416–427, Feb. 2002.

[16] A. György and T. Linder, "Codecell convexity in optimal entropy-constrained vector quantization," *IEEE Transactions on Information Theory*, vol. IT-49, no. 7, pp. 1821–1828, Jul. 2003.

[17] S. Dumitrescu and X. Wu, "On global optimality of gradient descent algorithms for fixed-rate scalar multiple description quantizer design," in *Data Compression Conference, 2005. Proceedings. DCC 2005*, Mar. 2005, pp. 388–397.

[18] Y. Linde, A. Buzo, and R. M. Gray, "An algorithm for vector quantizer design," *IEEE Transactions on Communications*, vol. COM-28, pp. 84–95, Jan. 1980.

[19] A. Gersho and R. M. Gray, *Vector Quantization and Signal Compression*. Boston, MA: Kluwer, 1992.

[20] T. Linder, "Learning-theoretic methods in vector quantization," in *Principles of Nonparametric Learning*, ser. CISM Courses and Lectures, L. Györfi, Ed. Wien, New York: Springer-Verlag, Sep. 2002, no. 434, ch. 4, pp. 163–210.

[21] A. György, T. Linder, P. A. Chou, and B. J. Betts, "Do optimal entropy-constrained quantizers have a finite or infinite number of codewords?" *IEEE Transactions on Information Theory*, vol. IT-49, no. 11, pp. 3031–3037, Nov. 2003.

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