Ph.D. THESIS SUMMARY

Optimal pebbling number of graphs

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Graph pebbling is a game on graphs. It was suggested by Saks and Lagarias to solve a number theoretic problem asked by Erdős, which was done by Chung. A *pebble distribution* P on a graph G is a function mapping the vertex set to nonnegative integers. We can imagine that each vertex v has $P(v)$ pebbles. The size of a pebble distribution P is the total number of pebbles, which we denote by $|P|$.

A *pebbling move* removes two pebbles from a vertex, which has at least two pebbles, and places one to an adjacent vertex. We say that a vertex v is *reachable* under the distribution P if there is a sequence of pebbling moves σ, such that v has at least one pebble after the execution of σ.

A pebble distribution P on G is *solvable* if all vertices of G are reachable under P. A pebble distribution on G is *optimal* if it is solvable and its size is minimal among all of the solvable distributions of G. The size of an optimal distribution is called *the optimal pebbling* number and denoted by $\pi_{\mathrm{opt}}(G)$.

Milans and Clark showed that if a graph G and an integer k is given, then deciding whether $\pi_{\mathrm{opt}}(G) \leq k$ is NP-complete.

There is another version of pebbling called rubbling. A *strict rubbling move* removes two pebbles from two distinct vertices and places one pebble at a common neighbor. A rubbling move is either a pebbling move or a strict rubbling move. If we replace pebbling moves with rubbling moves everywhere in the definition of the optimal pebbling number, then we obtain the *optimal rubbling number*, which is denoted by $\rho_{\mathrm{opt}}(G)$.

It is easy to see that $\pi_{\mathrm{opt}}(G) \leq 2^{\mathrm{diam}(G)}$. Muntz *et al.* claimed that for any integer k there is a diameter k graph G whose optimal pebbling number is $2^{\mathrm{diam}(G)}$. First we show that the original proof of this statement is incorrect. We prove the analogous statement for rubbling: for any integer k there is a diamater k graph G such that $\rho_{\mathrm{opt}}(G) = 2^{\mathrm{diam}(G)}$. We give a new simple proof for the pebbling case as well. To do this we use the distance-k domination number γ_k and we prove the following results: $\pi_{\mathrm{opt}}(G) \geq \rho_{\mathrm{opt}}(G) \geq \min\left(\gamma_{k-1}(G), 2^k\right)$ and $\pi_{\mathrm{opt}}(G) \geq \min\left(2^k, \gamma_{k-1}(G) + 2^{k-2} + 1, \gamma_{k-2}(G) + 1\right)$.

We show that for any $\epsilon > 0$ there is a graph G such that $\pi_{\mathrm{opt}}(G) \geq \frac{(4-\epsilon)n}{\delta+1}$, where δ is the minimum degree of G and n is the order of G. We prove that if $\mathrm{diam}(G) \geq 3$, then $\pi_{\mathrm{opt}}(G) \leq \frac{15n}{4(\delta+1)}$. We construct a family of graphs whose diameter can be arbitrarily large and their optimal pebbling number is at least $\left(\frac{8}{3} - \epsilon\right) \frac{n}{(\delta+1)}$. Finally we answer a question asked by Bunde *et al.*: "How large can $\pi_{\mathrm{opt}}(G)$ be when we require minimum degree δ?". The answer is that it can be as close to $\frac{4n}{\delta+1}$ as you wish but it cannot be reached.

We invent a method which can be used to give a lower bound on the optimal pebbling number of any vertex-transitive graph. Let $SG_{m,n}$ denote the m by n square grid graph. We prove that $\frac{2}{13}mn \leq \pi_{\mathrm{opt}}(SG_{m,n}) \leq \frac{2}{7}mn + O(m+n)$. We conjecture that the upper bound is strict. We define some induced subgraphs of $SG_{m,n}$ which we call staircase graphs. We determine the optimal pebbling number of the narrow staircases. The obtained values support our conjecture on $\pi_{\mathrm{opt}}(SG_{m,n})$.

A pebble distribution is called *t-restricted* if no vertex has more than t pebbles. The *t-restricted* optimal pebbling number of G, $\pi_t^*(G)$, is the size of the solvable t-restricted distribution of G containing the least number of pebbles. We prove that $\pi_{\mathrm{opt}}(G) = \pi_{\mathrm{opt}}(G \cdot K_m) = \pi_t^*(G \cdot K_m)$ if $t \geq 2$, where $\cdot$ denotes the lexicographic graph product. We use this to show that deciding whether $\pi_t^*(G) \leq k$ is NP-complete. We prove that if $\delta(G) \geq \frac{2}{3}n - 1$ then $\pi_2^*(G) = \pi_{\mathrm{opt}}(G)$. We show that there are infinitely many graphs, that satisfy $\delta < n/2 - 2$ and $\pi_{\mathrm{opt}}(G) \neq \pi_2^*(G)$.