Ph.D. THESIS OUTLINE

Optimal pebbling number of graphs

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1 Introduction

We denote the vertex set and the edge set of graph G by $V(G)$ and $E(G)$, respectively.

Graph pebbling is a game on graphs. It was suggested by Saks and Lagarias to solve a number theoretic problem asked by Erdős, which was done by Chung in 1989 [12].

A *pebble distribution* P on a graph G is a function mapping the vertex set to nonnegative integers. We can imagine that each vertex v has $P(v)$ pebbles. The size of a pebble distribution P is $\sum_{v\in V(G)} P(v)$ which we denote by $|P|$.

A *pebbling move* removes two pebbles from a vertex v and places one to an adjacent vertex u. A pebbling move is *allowed* if and only if the vertex loosing pebbles has at least two pebbles. A sequence of pebbling moves is called *executable* if for any i the ith move is allowed under the distribution obtained by the application of the first $i-1$ move.

We say that a vertex v is *k-reachable* under the distribution P if there is an executable pebbling sequence σ, such that v has at least k pebbles after the execution of σ. If $k = 1$, we say simply that v is reachable under P.

A pebbling distribution P on G is *solvable* if all vertices of G are reachable under P. A pebbling distribution on G is *optimal* if it is solvable and its size is minimal among all of the solvable distributions of G. The size of an optimal pebble distribution is called *the optimal pebbling* number and denoted by $\pi_{\mathrm{opt}}(G)$.

The optimal pebbling number was first mentioned in the paper of Patcher *et al.* [27] in 1995. Pebbling can be viewed as a transportation of resources problem. We can think of the pebbles as fuel containers. Then the loss of the pebble during a move is the cost of transportation. In case of optimal pebbling we are looking for an optimal assignment of fuel containers to the vertices such that any vertex can receive a container in case of need.

The optimal pebbling number of several graph families are known. For example exact values were given for paths and cycles [9, 17, 27], ladders [9], caterpillars [15] and m-ary trees [16]. There are also some known bounds on the optimal pebbling number. One of the earliest is that $\pi_{\mathrm{opt}}(G) \le 2^{\mathrm{diam}(G)}$ [26]. Bunde *et al.* investigated the connection between the optimal pebbling number and the minimum degree of the graph. They showed that $\pi_{\mathrm{opt}}(G) \le \frac{4n}{\delta+1}$ [9], where δ is the minimum degree of G. They also presented a construction for an infinity family of graphs with optimal pebbling number $(2.4 - \frac{24}{5\delta+15} - o(\frac{1}{n}))\frac{n}{\delta+1}$ [9].

If a graph G, a pebble distribution P on G and a target vertex v is given, then deciding whether v is reachable under P is NP-complete [24]. Deciding whether $\pi_{\mathrm{opt}}(G) \le k$ is also NP-complete [24].

In [8] the authors invented a version of pebbling called rubbling. A *strict rubbling move* removes two pebbles from two distinct vertices and places one pebble at a common neighbor. Thus a strict rubbling move is allowed if it removes pebbles from vertices who share a neighbor and both of them has a pebble. A rubbling move is either a pebbling move or a strict rubbling move. If we replace pebbling moves with rubbling moves everywhere in the definition of the optimal pebbling number, then we obtain the *optimal rubbling number*, which is denoted by $\rho_{\mathrm{opt}}(G)$. There are fewer results about rubbling than pebbling, we know four papers in the field of optimal rubbling [3, 6, 8, 23].

Researchers have been investigating the so called (capacity) restricted optimal pebbling in the last five years. A pebble distribution is called *t-restricted* if no vertex has more than t pebbles. The *t-restricted* optimal pebbling number of a graph G, denoted by $\pi_t^*(G)$, is the size of the solvable t-restricted distribution of G containing the least number of pebbles. This graph parameter is defined in [11]. In that paper the authors showed that $\pi_2^*(P_n) = \lceil 2n/3 \rceil$, where P_n is the n-vertex path, and they gave several upper bounds on $\pi_2^*(G)$ by using different domination parameters of G. In [29] Shiue showed that $\pi_2^*(T) = \lceil 2n/3 \rceil$ if T is an n-vertex tree.

We write $G \square H$ for the Cartesian product of graphs G and H. The vertex set of graph $G \square H$ is $V(G) \times V(H)$ and vertices (g, h) and (g', h') are adjacent if and only if either $g = g'$ and $\{h, h'\} \in E(H)$, or $h = h'$ and $\{g, g'\} \in E(G)$. We use $G^{\square d}$ as an abbreviation for $G \square G \square \cdots \square G$, where G appears exactly d times.

2 Optimal pebbling and rubbling of graphs with given diameter

The distance between vertices u and v, denoted by $d(u, v)$, is the number of edges contained in a shortest path connecting u and v. The diameter of graph G is the biggest distance in G. We use $\operatorname{diam}(G)$ to denote this parameter.

Placing $2^{\operatorname{diam}(G)}$ pebbles to a single vertex always creates a solvable distribution. This implies that $\pi_{\mathrm{opt}}(G) \leq 2^{\operatorname{diam}(G)}$, but usually much fewer pebbles are enough to construct a solvable distribution. It is natural to ask if there are graphs with arbitrarily large diameter where this amount of pebbles is required for optimal pebbling?

This question was investigated in [26] for the first time. The authors claimed that the answer is positive. However, their proof is incorrect. They gave an iterative construction of graphs, whose optimal pebbling number was believed to be two to the diameter. They claimed in [26], that if G is a graph with diameter d whose optimal pebbling number is 2^d, then $G \square K_{2^d+1}$ is a graph with diameter $d+1$ and optimal pebbling number 2^{d+1}. It is easy to see that $\operatorname{diam}(G \square K_{2^d+1}) = d + 1$, however its optimal pebbling number is not necessarily $2\pi_{\mathrm{opt}}(G)$.

Muntz *et al.* choose K_3 as a starting graph in their construction. The third graph in the sequence is $K_3 \square K_3 \square K_5$. We have created a solvable pebble distribution on this graph whose size is only 6. This is less than 8, what the authors claimed.

Herscovici *et al.* in [21] proved that $\pi_{\mathrm{opt}}(K_m^{\square d}) = 2^d$ if $m > 2^{d-1}$. In fact, a more general statement is proved in [21], but this is enough for our purposes. The diameter of these graphs is d, therefore they prove the sharpness of the diameter bound.

We can ask, what happens when we consider rubbling instead of pebbling? Unfortunately the proof of Herscovici *et al.* rely on several phenomena true for pebbling but false for rubbling. We answer this question and prove that $\rho_{\mathrm{opt}}(K_m^{\square d}) = 2^d$ if $m \geq 2^d$. Since $\rho_{\mathrm{opt}}(G) \leq \pi_{\mathrm{opt}}(G)$, this gives a short proof for the pebbling case as well. To do this we find a lower bound on the optimal rubbling number by using the distance k domination number, which we define in the next paragraph.

A *distance k dominating set S* of a graph G is a subset of the vertex set such that for each vertex v there is an element s of S whose distance from v is at most k. The *distance k domination number* of graph G, denoted by $\gamma_k(G)$, is the size of the smallest distance k dominating set.

Theorem 2.2 (Győri, Katona, Papp [3]) *Let G be a connected graph and k be an integer greater than one. Then* $\rho_{\mathrm{opt}}(G) \geq \min\left(\gamma_{k-1}(G), 2^k\right)$.

We are free to choose k. The best bound is obtained when $\gamma_{k-1} \approx 2^k$. Let $\Sigma_{m,d}$ be the following graph: We choose an alphabet Σ of size m. The vertices of $\Sigma_{m,d}$ are the words over Σ of length d. Two vertices are adjacent if and only if the corresponding words differ only at one position, roughly speaking their Hamming distance is one. It is well known that $\Sigma_{m,d} \simeq K_m^{\square d}$. We use this coding theory approach because it is easy to determine the diameter and the distance k domination number of $K_m^{\square d}$ this way.

It is easy to see that $\mathrm{diam}(\Sigma_{m,d}) = d$: We have to change all d characters of the word `aa...a` to obtain `bb...b`. Each of the changes requires passing through an edge. We can obtain any word from any other by changing each character at most once. Hence $\mathrm{diam}(\Sigma_{m,d}) = d$.

The set containing all constant words over alphabet Σ with length d is a distance $d-1$ dominating set because it is enough to change at most $d-1$ characters of a d long word to obtain a constant one. The number of these words is m. If we consider only $m-1$ words, then there is always a word which differs from each of them at each position. Therefore $\gamma_{k-1}(\Sigma_{m,d}) = m$. Using this we obtain, that:

Theorem 2.6 (Győri, Katona, Papp [3]) *Both the optimal pebbling and optimal rubbling number of $K_m^{\square d}$ is 2^d if $m \geq 2^d$.*

By using several properties of graph pebbling, we can improve Theorem 2.2 to give the following better bound on the optimal pebbling number:

Theorem 2.9 (Győri, Katona, Papp [3]) *For all $k \geq 3$ and any connected graph G whose order is at least two: $\pi_{\mathrm{opt}}(G) \geq \min\big(2^k, \gamma_{k-1}(G) + 2^{k-2} + 1, \gamma_{k-2}(G) + 1\big)$.*

We use Theorem 2.9 to determine the optimal pebbling number of $K_3 \square K_3 \square K_5$ which is exactly 6.

3 Optimal pebbling of graphs with given minimum degree

In this chapter we study the optimal pebbling of graphs with fixed minimum degree δ and we improve some results of [9]. We prove that there are infinitely many diameter two graphs whose optimal pebbling number is close to the $\frac{4n}{\delta+1}$ upper bound. More precisely:

Theorem 3.3 (Czygrinow, Hurlbert, Katona, Papp [1]) *For any $\epsilon > 0$ there is a diameter two graph G on n vertices with $\pi_{\mathrm{opt}}(G) > \frac{(4-\epsilon)n}{\delta+1}$.*

One may ask what happens if we consider larger diameter? In the second part of the chapter we construct a family of graphs with arbitrary large diameter, fixed minimum degree, and high optimal pebbling number. We determine the optimal pebbling number of the constructed graphs by using the collapsing technique, which is invented in [9].

Theorem 3.14 (Czygrinow, Hurlbert, Katona, Papp [1]) *For any $\epsilon > 0$ and any integer d, there is a graph G such that its diameter is greater than d and $\pi_{\mathrm{opt}}(G) \geq (\frac{8}{3} - \epsilon)\frac{n}{\delta+1}$.*

In the case when the diameter is at least three we also prove a stronger upper bound on the optimal pebbling number.

Theorem 3.15 (Czygrinow, Hurlbert, Katona, Papp [1]) *Let G be a connected graph having diameter at least 3 and with minimum degree δ. Then we have $\pi_{\mathrm{opt}}(G) \leq \frac{15n}{4(\delta+1)}$.*

We do this by showing the existance of a solvable pebble distribution whose size is not too big. We use the following new definition during the proof:

Definition 3.17 *A vertex $v \in V(G)$ is* strongly reachable *under the pebble distribution D if v and all of its neighbors are reachable under D.*

We give an algorithm to show that there is an initial distribution D_0 whose size is at most $4(\delta+1)/15$ times the number of strongly reachable vertices. Then we show that we can extend D_0 by adding more pebbles and keeping this ratio not bigger than $4(\delta+1)/15$, until the obtained distribution is solvable. Note that all vertices are strongly reachable under a solvable distribution.

We use Theorem 3.15 to show the following:

Claim 3.31 (Czygrinow, Hurlbert, Katona, Papp [1]) *There is no connected graph G such that $\pi_{\text{opt}}(G) = \frac{4n}{\delta+1}$.*

We can combine this claim with Theorem 3.3 to answer a question asked in [9], which was "How large can $\pi_{\text{opt}}(G)$ be when we require minimum degree δ?"

Corollary 3.32 (Czygrinow, Hurlbert, Katona, Papp [1]) *For any graph G we have $\pi_{\text{opt}}(G) < \frac{4n}{\delta+1}$, and this bound is sharp.*

4 Staircase graphs

We denote the n by m square grid by $SG_{n,m} \cong P_n \square P_m$. The optimal pebbling number of grids has been investigated by many authors. Exact values were proved for $P_n \square P_2$ [9] and $P_n \square P_3$ [33]. The question for bigger grids is still open. We gave a construction [2], which can be seen in Figure 1. This construction gives the following theorem:

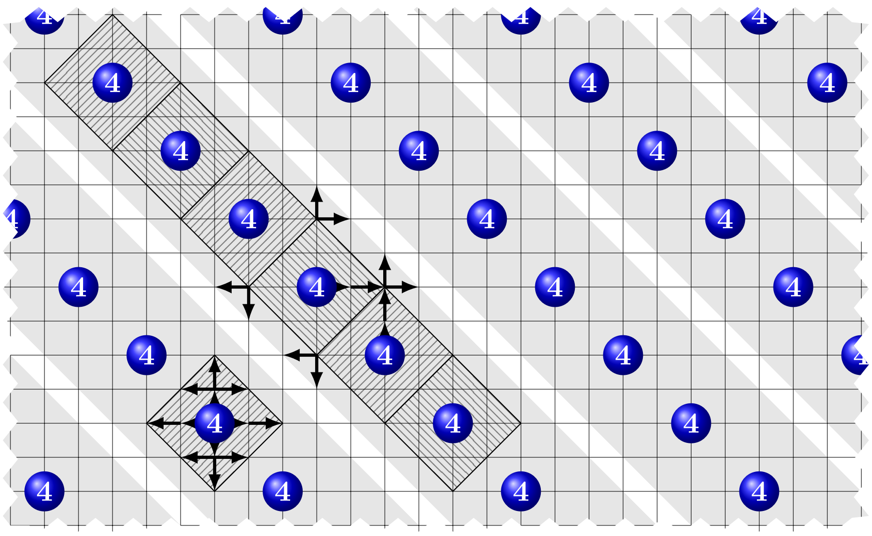

Figure 1: Solvable distribution of the square grid.

Theorem 4.1 (Győri, Katona, Papp [2]) $\pi_{\text{opt}}(SG_{n,m}) \leq \frac{2}{7}nm + O(n+m) \approx 0.2857nm + O(n+m)$

The distribution P which we have constructed takes groups of seven consecutive diagonals and places pebbles on the middle one (see Figure 1). Using these pebbles, it is possible to reach any vertex on any diagonal in the group.

We conjecture that P is an optimal pebble distribution on the square grid graph, however we do not know a proof for this. Can we at least show that the distributions induced by P on these induced subgraphs containing seven consecutive diagonals are optimal? If this was not the case, it would refute the conjecture. These considerations provide the main motivation for this chapter.

We investigate a family of graphs which we call *staircase graphs*. These graphs are connected induced subgraphs of the square grid. The width seven instances correspond to the groups of seven diagonals discussed above.

Let $SG = P_\infty \,\square\, P_\infty$ be the infinite square grid where P_∞ is the doubly infinite path with vertex set $\mathbb{Z}$ and edge set $\{\{i, i+1\} : i \in \mathbb{Z}\}$.

Definition 4.2 *For any $k \in \mathbb{Z}$, we say that $D_k^+ = \{\{i,j\} \in V(SG) : i - j = k\}$ is a* positive diagonal *of SG. Similarly we define the* negative diagonal*: $D_k^- = \{\{i,j\} \in V(SG) : i + j = k\}$.*

A staircase graph will be defined in terms of the intersection of a set of consecutive positive diagonals in SG with a set of consecutive negative diagonals. When the number of diagonals taken in each direction is odd, there will be two nonisomorphic graphs to consider. For examples see Figure 2.

Definition 4.3 *For odd m, let $S'_{m,n}$ be the graph induced by the vertex set $\left(\cup_{j=1}^m D_j^-\right) \cap \left(\cup_{i=1}^n D_i^+\right)$, and let $S_{m,n}$ be the graph induced by $\left(\cup_{j=1}^m D_j^-\right) \cap \left(\cup_{i=0}^{n-1} D_i^+\right)$.*

For even m, let $S_{m,n}$ be the graph induced by the vertex set $\left(\cup_{j=1}^m D_j^-\right) \cap \left(\cup_{i=1}^n D_i^+\right)$. In this case we have only one isomorphism class.

Note that $S'_{m,n} \cong S_{m,n}$ if n is even. We say that m and n are the *width* and the *length* of the staircase graph, respectively, and generally assume that $n \geq m$. We will refer to the graphs $S_{m,n}$ and $S'_{m,n}$ as *m-wide staircase graphs*.

A 1-wide staircase graph is an edgeless graph, therefore its optimal pebbling number equal to its order. A 2-wide staircase graph is a path, thus $\pi_{\text{opt}}(S_{2,n}) = \pi_{\text{opt}}(S'_{2,n}) = \pi_{\text{opt}}(P_n) = \left\lceil \frac{2n}{3} \right\rceil$.

We use the collapsing technique [9] again with induction to prove our results for staircase graphs. First we use it for narrow staircase graphs and extend it for wider ones.

Theorem 4.4 (Győri, Katona, Papp, Tompkins [5]) *If $4k + r \geq 2$ where $k \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3\}$, then*

$$\pi_{\text{opt}}(S_{3,4k+r}) = 3k + r,$$
$$\pi_{\text{opt}}(S'_{3,4k+r}) = \begin{cases} 3k+2 & \text{if } r = 3 \\ 3k + r & \text{otherwise.} \end{cases}$$

Theorem 4.9 (Győri, Katona, Papp, Tompkins [5]) $\pi_{\text{opt}}(S_{4,4k+r}) = 3k + r$ *except for $n \in \{1, 2\}$.* $\pi_{\text{opt}}(S_{4,1}) = 2, \pi_{\text{opt}}(S_{4,2}) = 3$.

Theorem 4.10 (Győri, Katona, Papp, Tompkins [5]) $\pi_{\text{opt}}(S_{5,5k+r}) = \pi_{\text{opt}}(S'_{5,5k+r}) = 4k + r$, *except for $n \in \{1, 2, 3, 7\}$.* $\pi_{\text{opt}}(S_{5,3}) = \pi_{\text{opt}}(S'_{5,3}) = 4$ *and* $\pi_{\text{opt}}(S'_{5,7}) = 7$.

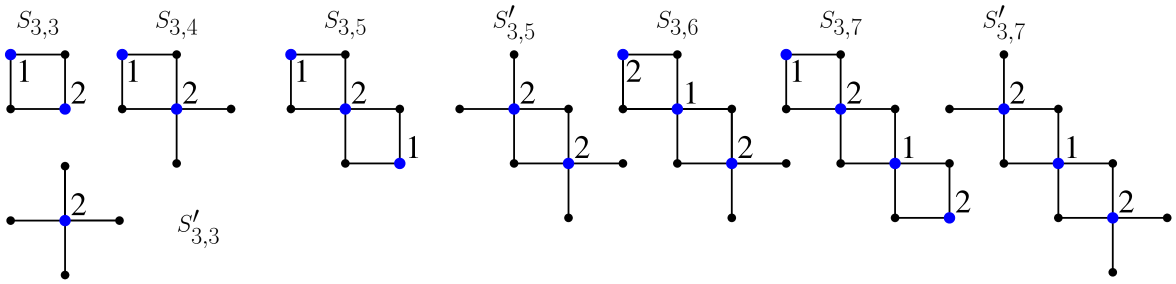

Figure 2: Optimal distributions of small $S_{3,n}$ and $S'_{3,n}$ graphs.

Theorem 4.13 (Győri, Katona, Papp, Tompkins [5]) *$\pi_{\text{opt}}(S_{6,n}) = n$, except for $n \in \{1, 2, 3, 4, 8, 9\}$. $\pi_{\text{opt}}(S_{6,3}) = \pi_{\text{opt}}(S_{6,4}) = 5$, $\pi_{\text{opt}}(S_{6,8}) = 9$ and $\pi_{\text{opt}}(S_{6,9}) = 10$.*

Theorem 4.16 (Győri, Katona, Papp, Tompkins [5]) *If $S^*_{7,n}$ is $S_{7,n}$ or $S'_{7,n}$, then*

$$n + 1 \leq \pi_{\text{opt}}(S^*_{7,n}) \leq n + 3.$$

The lower bound is sharp for graphs $S_{7,5}$, $S_{7,6}$, $S_{7,7}$, $S_{7,8}$ and every $S'_{7,n}$ where $n \equiv 3 \bmod 4$.

Unfortunately, we could not determine the exact value of $\pi_{\text{opt}}(S^*_{7,n})$. We cannot prove $n + 2$ as a lower bound by the collapsing technique, but for some $S^*_{7,n}$ graphs we have not found a solvable pebble distribution using $n + 1$ pebbles.

A natural question that arises is: what is the optimal pebbling number of $S_{8,n}$? We determined the values when $n \leq 7$, but we think that the general behavior of the eight-wide case differs from the seven-wide case. We obtained $\pi_{\text{opt}}(S_{8,8}) = 11$ by solving an integer program. We used a computer for this task. Unfortunately, even the $n = 9$ case requires more computational power than an average PC has. We have found some solvable distributions which use approximately $5n/4$ pebbles. We conjecture that $\pi_{\text{opt}}(S_{8,n}) = \frac{5}{4}n + O(1)$.

5 A lower bound on the optimal pebbling number of the square grid

Instead of the square grid on the plane it is easier to work with the square grid on the torus. Note that the m by n torus grid $T_{m,n}$ is isomorphic to $C_m \Box C_n$, where C_n is the n-vertex cycle. As the plane grid is a subgraph of this, any lower bound on the torus grid will give a lower bound on the plane grid as well. It is well known that the torus grid is a *vertex-transitive* graph, i.e. given any two vertices v_1 and v_2 of G, there is some automorphism $f : V(G) \to V(G)$ such that $f(v_1) = v_2$.

In this chapter we present a new method giving a lower bound on the optimal pebbling number of vertex-transitive graphs. The method is a bit complicated, it requires a lot of definitions. We use the concept of excess, which was introduced in [33], but we need to introduce many new definitions as well. The basic definition of excess is the following:

Definition 5.1 *Let $\text{Reach}(P, v)$ be the greatest integer k such that v is k-reachable under P. The* excess *$\text{Exc}(P, v)$ of v under P is $\text{Reach}(P, v) - 1$ if v is reachable and zero otherwise. Let $TE(P)$ denote the total excess, so $TE(P) = \sum_{v \in V(G)} \text{Exc}(P, v)$*

The *distance-k open neighborhood* of a vertex v, denoted $N^k(v)$, consists of all vertices whose distance from v is exactly k.

Definition 5.2 *The* effect *of a pebble placed at v is* $\mathrm{ef}(v) = \sum_{i=0}^{\mathrm{diam}(G)} \left(\frac{1}{2}\right)^i |N^i(v)|$.

If the graph is vertex-transitive, then $\mathrm{ef}(v)$ is the same for each vertex v. Herscovici *et al.* [20] proved that if G is a vertex-transitive graph, then $|V(G)|/\,\mathrm{ef}(v)$ is a lower bound on the optimal pebbling number of G. We improve this result:

Theorem 5.3 (Győri, Katona, Papp [4]) *If P is a solvable distribution on G, then*

$$\sum_{v \in V(G)} \mathrm{ef}(v)P(v) \geq |V(G)| + TE(P).$$

It is easy to see that in an optimal distribution many vertices can have more than one pebble after the execution of some pebbling moves. Therefore the total excess is high. Note that the distribution containing one pebble at each vertex is solvable but it has 0 excess, on the other hand that distribution usually contains much more pebbles than an optimal distribution. This shows that the tool called excess itself is not enough to improve the lower bound. We invent several other notions but we omit most of them from this outline. We define only those ones which are required to state Theorem 5.45.

Definition 5.6 *The* coverage *of a distribution P is the set of vertices which are reachable under P. We denote the size of this set with* $\mathrm{Cov}(P)$.

Definition 5.9 *We say that a distribution U is a* unit*, if all the pebbles are on a single vertex.*

Units are the building blocks of pebble distributions in the following sense: Any distribution P can be written as $\sum_{u|P(u)>0} P_u$, where P_u is a unit having $P(u)$ pebbles at u. The set $\{P_u|P(u) > 0\}$ is called as the disjoint decomposition of P to unit distributions. Units have two main advantages over other distributions. Their coverage and total excess can be easily calculated:

Claim 5.10 (Győri, Katona, Papp [4]) *Let U be a unit distribution which places pebbles at vertex u. Then we have that*

$$\mathrm{Cov}(U) = \sum_{i=0}^{\lfloor \log_2(U(u)) \rfloor} |N^i(u)|,$$

$$\mathrm{TE}(U) = \sum_{i=0}^{\lfloor \log_2(U(u)) \rfloor} |N^i(u)| \left(\left\lfloor \frac{U(u)}{2^i} \right\rfloor - 1 \right).$$

Now we can state our result which gives a lower bound on any vertex-transitive graph.

Corollary 5.45 (Győri, Katona, Papp [4]) *If P is a solvable distribution on a vertex-transitive graph G, v is a random vertex of G, Δ denotes the degree of v and $\{U_1, U_2, \ldots, U_t\}$ is a disjoint decomposition of P to unit distributions, then*

$$|P| \geq \frac{\frac{\Delta-1}{\Delta-2}|V(G)| + \sum_{i=1}^{t} TE(U_i) - \frac{1}{\Delta-2}\sum_{i=1}^{t} \mathrm{Cov}(U_i)}{\mathrm{ef}(v)}$$

It is easy to calculate that $\mathrm{ef}(v) < 9$ in $T_{m,n}$. After calculating some bounds on $TE(U_i)$ and $\mathrm{Cov}(U_i)$ we can deduce the following result:

Theorem 5.49 (Győri, Katona, Papp [4]) *The optimal pebbling number of $T_{m,n}$ is at least $\frac{2}{13}nm$, when $m, n \geq 5$.*

We can obtain $SG_{m,n}$ from $T_{m,n}$ by edge deletion, therefore $\pi_{\mathrm{opt}}(SG_{m,n}) \geq T_{m,n}$. This gives our main result of the chapter:

Corollary 5.50 (Győri, Katona, Papp [4]) *The optimal pebbling number of $SG_{n,m}$ is at least $\frac{2}{13}nm$ when $n, m \geq 5$.*

We also get a new proof for $\pi_{\mathrm{opt}}(P_n) = \pi_{\mathrm{opt}}(C_n) = \lceil 2n/3 \rceil$ as a byproduct of Theorem 5.45.

6 Restricted optimal pebbling

It is easy to see that $\pi_2^*(G) \geq \pi_t^*(G) \geq \pi_{t+1}^*(G) \geq \pi_{\mathrm{opt}}(G)$. It is an interesting question: What graphs 2-restricted optimal pebbling number and optimal pebbling number are the same. Our first result in this topic requires the definition of the lexicographic graph product.

Definition 6.2 *$G \cdot H$ denotes the lexicographic product of graphs G and H. It is defined as follows: $V(G \cdot H) = V(G) \times V(H)$ and (g_1, h_1) and (g_2, h_2) are adjacent iff either $\{g_1, g_2\} \in E(G)$ or $g_1 = g_2$ and $\{h_1, h_2\} \in E(H)$.*

Theorem 6.4 (Papp [7]) *If G is a connected n-vertex graph, $m \geq \left\lceil \frac{n}{3} \right\rceil$ and $t \geq 2$, then*
$\pi_{\mathrm{opt}}(G) = \pi_{\mathrm{opt}}(G \cdot K_m) = \pi_t^*(G \cdot K_m)$.

We use this theorem to show that the calculation of t-restricted optimal pebbling numbers is computationally hard. We consider two decision problems:

OPN:
Instance: a graph G and an integer k:
Question: is $\pi_{\mathrm{opt}}(G) \leq k$?

ROPN:
Instance: a graph G and integers $t \geq 2$, k:
Question: is $\pi_t^*(G) \leq k$?

Milans and Clark proved that OPN is NP-complete [24]. The previous theorem naturally gives us a Karp reduction OPN$\prec$ROPN. ROPN is in NP, therefore our main complexity result is:

Theorem 6.6 (Papp [7]) *ROPN is NP-complete.*

The authors of [11] asked for a characterization of graphs whose optimal pebbling number and 2-restricted optimal pebbling number is the same. We believe that such a characterization is elaborate. Note that there are many graphs which belong to that class. For example paths, cycles and complete graphs. We investigate the question of what value of the minimum degree guarantees that $\pi_{\mathrm{opt}}(G) = \pi_2^*(G)$. We prove a sufficient condition:

Claim 6.9 (Papp [7]) *Let G be an n-vertex graph. If $\delta(G) \geq \frac{2}{3}n - 1$, then $\pi_2^*(G) = \pi_{\text{opt}}(G)$.*

We also show that if the minimum degree is less than $n/2 - 2$, then there are infinitely many graphs for which these two parameters have different values.

List of my Publications

[1] A. Czygrinow, G. Hurlbert, Gy. Y. Katona, L. F. Papp, Optimal pebbling number of graphs with given minimum degree *Discrete Applied Mathematics*, **260** (2019) pp. 117–130.

[2] E. Győri, G. Y. Katona, L. F. Papp, Constructions for the optimal pebbling of grids, *Periodica Polytechnica Electrical Engineering and Computer Science* **61 no. 2** (2017) pp. 217–223.

[3] E. Győri, Gy. Y. Katona, L. F. Papp, Optimal pebbling and rubbling of graphs with given diameter *Discrete Applied Mathematics*, **266** (2019) pp. 340–345.

[4] E. Győri, G. Y. Katona, L. F. Papp, Optimal Pebbling Number of the Square Grid *Graphs and Combinatorics* **36** (2020) pp. 803–829.

[5] E. Győri, G. Y. Katona, L. F. Papp, C. Tompkins, Optimal Pebbling Number of Staircase Graphs, *Discrete Mathematics* **342** (2019) pp. 2148–2157.

[6] G. Y. Katona, L. F. Papp, The optimal rubbling number of ladders, prisms and Möbius-ladders, *Discrete Applied Mathematics* **209** (2016) pp. 227–246.

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